Net contribution, Liquidity, and Optimal Pension Management

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Abstract

This paper presents an optimal portfolio balancing strategy in discrete time for a CRRA investor, such as a pension fund, who invests only on one risk-free asset and one risky asset where both fixed and (linear) proportional transaction costs exist. Based on our theoretical results, we provide a heuristic that can generate an approximate solution to this problem while considering periodic (negative or positive) changes in net contribution, which occurs often for pension funds. According to our computational results, our optimal asset allocation strategies match actual asset allocation schemes of some internationally renowned pension funds. Furthermore, we also learned that net contribution and liquidity have significant impacts on an optimal asset allocation of a pension fund.

1 Introduction

While experiencing one of the most severe crises in the financial market around 2008 since the Great Depression, it has been noticed that roughly there are two groups of internationally renowned pension service providers (PSPs). The first group has not altered the asset allocation considerably while the other has been actively adjusted the mix of their portfolios so as to adapt into adverse market conditions. This paper is motivated by this observation and focuses on investment strategies of these PSPs. Specifically this research aims to come up with theoretical asset allocation schemes to examine the optimal investment strategy of a few large pension funds in the world.

In this paper, we suggest an optimal portfolio balancing framework in discrete time for pension funds investing only in one risk-free asset and one risky asset paying both proportional and fixed transaction costs. We also provide multi-period optimal portfolio balancing strategies while considering periodic (negative or positive) cash flows into funds at the beginning of each period. For pension funds, these cash flows are usually the net contribution: the difference between the contributions from participating members and the annuity (benefits) payments to members. This research also examine the impact of market liquidity on the optimal asset portfolio strategies of large pension funds using the bid-ask spreads and the proportional transaction cost variations, even though bid-ask spread could understate the true transaction costs for large investors like pension funds (Marshall (2006)).

Portfolio optimization is one of the most studied topics in finance. Following the footsteps of seminal papers Merton (1969, 1971), Cvitanic and Karatzas (1992), Xu and Shreve (1992), and He and Pearson (1993) considered portfolio optimization problems with some constraints on the strategies. Davis and Norman (1990), Magill and Constantinides (1976), and Taksar, Klass, and Assaf (1988) considered a proportional transaction cost to make the problem more realistic. Constantinides (1979) showed that no-trading region is a convex cone for an investor with a power utility and a proportional transaction cost. Later, Constantinides (1986) provided an approximate solution to this problem. Eastham and Hastings (1988) made portfolio optimization problem more realistic by considering both fixed and proportional transaction costs in their impulse control approach. Korn (1998) further improved the work of Eastham and Hastings (1988) by utilizing an optimal stopping criteria method. Gennotte and Jung (1994) numerically identified the approximate boundary values of no-trading region for an portfolio optimization problem with a finite terminal date.

While studies regarding optimal portfolio rebalancing in continuous-time abound, the research on portfolio rebalancing in discrete-time is scarce although it has more practical applications. Boyle and Lin (1997) presented a closed-form solution to the finite horizon problem when there is a proportional transaction cost. (Let us refer to this problem as PTC.) We add a fixed transaction cost factor in the model and compare our work to the result of Boyle and Lin (1997) showing how the shape of no-trading region changes. In a continuous time model, such as Korn (1998), a fixed transaction cost can be easily incorporated because it can be assumed that a fixed transaction cost is incurred whenever an investor tries to rebalance his/her portfolio.

However, incorporating a fixed transaction cost into a portfolio optimization problem in discrete time is more complicated because an investor may or may not rebalance his/her portfolio at a certain period, which makes a fixed cost as a binary variable. When a binary variable exists, a terminal wealth maximization problem comes to have a discontinuous point where the expected terminal wealth function is not differentiable. We tackle this problem by separating the problem into two parts: one that assumes no trading and the other that assumes a positive amount of trading a risky asset. In addition to presenting an explicit strategy for one-period optimal portfolio rebalancing problem with both fixed and proportional cost, we introduce a (brute-force) heuristic that finds an approximate solution for multi-period portfolio optimization problem by extending the one-period optimal strategy.

Another important factor that affects optimal asset allocation is liquidity of the risky asset (or stock market). According to the research on liquidity measures by Sarr and Lybek (2002), the types of liquidity measures include the followings; transaction cost measures, volume-based measures, price-based measures, market-impact measures, and other econometric techniques. According to their analysis, the main factors composing liquidity measures are bid-ask spreads, turnover ratios, and price impact measures.

Meanwhile, Plerou, Gopikrishnan, and Stanley (2005) analyzes how to explain the market liquidity, using bid-ask spread. Also, Wei and Zheng (2010) measure the liquidity of individual equity options by the bid-ask spread. Bid-ask spread can be expressed as the absolute value or the ratio of a gap between the bid price and ask price. For example, Marshall and Young (2003) calculated the bid-ask spread using every Wednesday's closing bid and ask prices. According to the research by Marshall (2006), the order based measure such as the bid-ask spread is effective for measuring the liquidity of small investors but, is not perfect for measuring the liquidity of larger investors. Marshall (2006) insists that weighted order value can make up for the weak points of traditional liquidity proxies by incorporating bid-ask spreads and market depth. Weighted order value is one of the liquidity proxies, already used by Aitken and Comerton (2003). This paper takes bid-ask spreads as a measure of the market liquidity into account and computes variations of proportional transaction cost according to the changes in bid-ask spreads. Moreover, it analyzes how the market liquidity affects optimal asset portfolio strategies.

The contribution of paper is summarized as follow. Technically, under discrete time frame work, this paper is the first research on the strategies of optimal asset portfolio, considering both fixed transaction cost and proportional transaction cost. (Let us refer to this problem as FPTC.) Using the frame work, the asset allocation strategies of a few world major pension funds are analyzed and compared while taking their net contributions, the gap between contributions paid by pension members and annuity payments spending for beneficiaries, into account. The changeable degree of liquidity according to market situation is also one of our considerations solving the optimal pension management problem. Surprisingly, this research indicates that actual investment strategies by a few large pension providers are very close to the theoretically optimal investment strategies suggested by this research, which implies that our model can be useful in terms of actual pension fund management.

This paper is divided into six sections. The research motivation and the basic information about the PSPs of our interest are provided in section 2. To solve the proposed research problems, our models are introduced in section 3. Using the model, optimal solutions are provided in section 4. First PTC problem is analyzed in detail in our own terms. Then, we extend the theoretical properties of PTC to come up with an optimal trading strategy for FPTC problem. The results of our research are displayed in the three subsections of section 5. subsection 5.1 discusses a heuristic (a recursive algorithm) that generates an approximate optimal solution for a multi-period FPTC problem. In subsection 5.2, we describe how changes in parameters affect the no-trading region (NTR) for the FPTC problem. We apply our asset allocation framework to several world renowned pension funds and compare our theoretical asset allocation schemes to their actual asset allocations in subsection 5.3. Finally, section 6 concludes the paper with a few suggestions for future research.

2 Motivation: PSPs of interest

While surveying investment strategies of world renowned PSPs, during the recent financial market crisis between 2008 and 2009 , we made an interesting observation. We discovered that some PSPs actively adjusted asset allocations while the others held onto their existing asset allocations schemes. Through efforts of delving into this issue, we learned that some factors, such as liquidity, transaction costs, and changes in net contribution, have significant impacts on PSPs asset allocation policies. In this paper we provide some answers for this problem while taking above-mentioned factors into account.

Information about PSPs of our interest is in Table 1 including their names, abbreviated names, base countries of their operations, estimated assets in billion USD and valuation dates of their assets.

Table 1: Analyzed PSPs

Name	Abbrev.	Country	Net asset (Mill. USD)	Valuation date
CalPERS	CALP	Unites States	241,761	Jun 2011
New York State and Local Retirement System	NYSLR	United States	149,548	March 2011
Avon Pension Fund	AVON	United Kingdom	4,098	March 2011
Första AP-fonden	FAPF	Sweden	33,883	December 2011
National Pension Service	NPS	South Korea	312,934	December 2011

Table 2: World Stock Market Indices

Year	Spread	Scaled	Volatility
2002	0.27	4.02	19.30 %
2003	0.18	2.66	12.26 %
2004	0.14	2.03	8.19 %
2005	0.15	2.15	8.19 %
2006	0.15	2.13	7.45 %
2007	0.16	2.36	9.33 %
2008	0.23	3.36	23.63 %
2009	0.09	1.36	23.27 %
2010	0.08	1.13	20.60 %
2011	0.07	1.00	17.97 %

Table 2 include world stock market indices such as bid-ask spread (Spread), Scaled spread(Scaled) and stock market volatility (or σ). We calculated 'Scaled' by dividing 'Spread' with the smallest spread between 2002 and 2011, which is 0.07 (Spread of year 2011). Since higher 'Spread' indicates higher trading cost, we will assume that trading cost increases as 'Scaled' does.

From Table 2, we can see that world stock market volatility varied between 7.45% and 23.64% between 2002$\sim$2011. To reflect volatility changes of stock market, we ran tests using different values of σ, which are 10%, 15% and 20%.

During the same period, 'Scaled' changed from 1 to 4.02. Assuming that larger spread is an indication of larger proportional trading cost, we tried 0.5%, 1.5% and 2.5% for possible values of k_1 and k_2. (Throughout this paper, we assume that k_1 and k_2 have the same value.) Again, as we stated in section 1, it is difficult to say that proportional trading cost increases in proportional to the spread and these parameter changes are tried so that we could test some probable cases.

Table 7$\sim$11, actual asset allocations and net cash flows from contributions and benefits of PSPs of our interest. Columns in these tables include their asset allocations in stocks (risk asset) and bonds(risk-free asset), members' contributions, and benefits payments.

Currency units for each table are CALP(thousand USD), NYSLR(thousand USD), AVON (thousand LB), FAPF(million SEK), and NPS(million KRW). In actual investment operation, there are no true risk-free assets.

In Tables 7$\sim$11, we considered all bonds as a risk-free asset and all stocks as a risky asset. Although some government treasury bills have little credit risk, it is still possible to lose money from these bonds when their market value drops. Yet, we considered all the bonds as risk-free assets because most PSPs usually invest only on bonds with decent credit ratings and their default ratio is very low.

Most asset values In Tables 7∼11 are market values. In these tables, we can see that there had been a big drop in asset values between 2008 and 2010 due to a financial market crash in 2008 and following economic crisis. During this period, asset values of 5 PSPs show common and distinct changes. One common change is the depreciation of risky asset values. In addition, most PSPs risk-free asset values also dropped with an exception of NPS, possibly due to a withdrawal rush. (See sudden drops in net contribution of CALP, NYSLR, and FAPF.) Decrease in risky asset values during this period can be explained by a few factors such as stock market crash and liquidation of stocks, which might have been mitigated by fund managers efforts to defend asset value depreciation.

Net contributions of these PSPs are CALP (around -2%), NYSLR (around -4%), AVON (from around negative 1% in 2000, became 1% recently), FAPF(from 2% in 2002, turned negative in 2009) and NPS(close to zero). As we mentioned above, some PSPs net contributions fluctuated as stock market crashed. However, net contribution of NPS showed almost no change (NPS membeship is mandatory in Korea) and AVON showed an increase in membership contributions since 2006.

3 The model

First, we assume that two classes of asset are traded in the financial market: a risky asset (or a stock) and a risk-free asset (or a bond). The one-period return of the risk-free asset is R. We assume there are two possible outcomes of one-period return of the risky asset, u and d $(0 < d < R < u)$ with probability π_u and π_d $(0 \leq \pi_u, \pi_d \leq 1,\ \pi_u + \pi_d = 1)$. Let z_t be the price process of the risky asset. We also assume the financial market has market frictions, hence, market participant pays fixed and proportional transaction costs when selling or purchasing the risky asset. The per-period fixed cost is denoted by K and the per-period proportional transaction costs for sales (purchases) is k_1 (k_2, respectively).

The investor considered in this paper is a pension fund manager who has the following constant relative risk aversion (CRRA)-type utility preference for wealth:

$$u(w) := \frac{w^{1-\gamma}}{1-\gamma}, \tag{1}$$

where γ is the coefficient of relative risk aversion and satisfies $\gamma > 0$ and $\gamma \neq 1$.

We denote x_t (y_t) as a pair of dollar amounts invested in the risk-free (risky, respectively) asset at time t $(t \in \{0, 1, \ldots, T\})$. We assume that positive amount of the risky asset can be traded at time t only when the portfolio at time $(t+1)$, (x_{t+1}, y_{t+1}), stays in a *solvency region*, which is defined as

$$\mathcal{S} := \{(x, y) |\ x + (1 - k_1)y - K \geq 0,\ y \geq 0\}.$$

Note that the first condition reflects the assumption that the liquidated wealth should be positive when $y > 0$ and the second stands for the short-sale constraint of the risky asset.

The fund manager considered in our paper want to maximize her expected utility for wealth at the final time T,

$$E[u(x_T + y_T)],$$

by controlling their investment amount in the risky asset. Here, $E[\cdot]$ is the expectation under the probabilities π_u and π_d.

We denote $\triangle_-$ ($\triangle_+$) as a dollar amount of the risky asset for sales (purchases, respectively) at time t. (For convenience sake, we do not mark time variable t when utilizing $\triangle_-$ and $\triangle_+$.) Then, we get the following relationships between asset holdings at time t and those at time $(t+1)$:

$$x_{t+1} = \begin{cases} R(x_t + (1-k_1)\triangle_- - K), \text{where } \triangle_- > 0 \\ R(x_t - (1+k_2)\triangle_+ - K), \text{where } \triangle_+ > 0 \\ Rx_t, \text{where } \triangle_- = 0 \text{ and } \triangle_+ = 0 \end{cases}$$

and

$$y_{t+1} = \begin{cases} (y_t - \triangle_-)z_t, \text{where } \triangle_- > 0 \\ (y_t + \triangle_+)z_t, \text{where } \triangle_+ > 0 \\ y_t, \text{where } \triangle_- = 0 \text{ and } \triangle_+ = 0. \end{cases}$$

4 Optimal pension fund management

In this section we show the one-period optimal trading strategy for our problem.

We define FPTC sell and buy problems at time k $(0 \le k < T)$ as follow:

$$\begin{array}{rl} \max & f_-(\triangle_-; x_k, y_k) \\ \text{s.t.} & \triangle_- \ge 0, (x_{k+1}, y_{k+1}) \in \mathcal{S}, \end{array} \qquad \begin{array}{rl} \max & f_+(\triangle_+; x_k, y_k) \\ \text{s.t.} & \triangle_+ \ge 0, (x_{k+1}, y_{k+1}) \in \mathcal{S}, \end{array}$$

where

$$\begin{aligned} f_-(\triangle_-; x_k, y_k) := \; & \pi_u u(x_k R + y_k u - KR + ((1-k_1)R - u)\triangle_-) \\ & + \pi_d u(x_k R + y_k d - KR + ((1-k_1)R - d)\triangle_-) \end{aligned} \tag{2}$$

and

$$\begin{aligned} f_+(\triangle_+; x_k, y_k) := \; & \pi_u u(x_k R + y_k u - KR + (-(1+k_2)R + u)\triangle_+) \\ & + \pi_d u(x_k R + y_k d - KR + (-(1+k_2)R + d)\triangle_+). \end{aligned} \tag{3}$$

Our approach for solving the one-period problem is to find the optimal strategies $\triangle_-^*$ and $\triangle_+^*$ for the sell and buy problems and pick the strategy which make the expected utility have maximum value. In other words, the optimal investment strategy of our one-period problem is the strategy which make us attain the maximum of

$$\max\{f_-(\triangle_-^*; x_0, y_0), f_+(\triangle_+^*; x_0, y_0), f_0(0; x_0, y_0)\}$$

where

$$f_0(0; x_k, y_k) := \pi_u u(x_k R + y_k u) + \pi_d u(x_k R + y_k d). \tag{4}$$

Notice that f_0 represent the expected utility where no transaction occurs at time k.

We know that PTC problem is FPTC problem where the fixed transaction cost K is zero. Let us define g_- (g_+ and g_0) as f_- (f_+ and f_0, respectively) where $K = 0$. As a result, the one-period PTC problem can be stated as the problem to find

$$\max\{g_-(\triangle_-^*; x_0, y_0), g_+(\triangle_+^*; x_0, y_0), g_0(0; x_0, y_0)\}.$$

Throughout this paper, we will often work with the sell-problem only. Analysis of the buy-problem will be omitted because it can be regarded as a mirror-image of the sell problem.

4.1 Analysis of PTC

First, we can check the optimal buy and sell strategies $\triangle_-^*$ and $\triangle_+^*$ satisfy

$$w_1 = \frac{x_k R + y_k u + ((1-k_1)R - u)\triangle_-^*}{x_k R + y_k d + ((1-k_1)R - d)\triangle_-^*} \tag{5}$$

and

$$w_2 = \frac{x_k R + y_k u + (-(1+k_2)R + u)\triangle_+^*}{x_k R + y_k d + (-(1+k_2)R + d)\triangle_+^*}, \tag{6}$$

where w_1 and w_2 are defined as

$$w_1 := \left(-\frac{\pi_u((1-k_1)R - u)}{\pi_d((1-k_1)R - d)}\right)^{\gamma^{-1}}, \quad w_2 := \left(-\frac{\pi_u((-1+k_2)R + u)}{\pi_d(-(1+k_2)R + d)}\right)^{\gamma^{-1}}. \tag{7}$$

Note that (5) and (6) are equations and (7) are definitions. The amounts w_1 and w_2 play crucial roles in our analyses. As you can see in (7), w_1 and w_2 are constants that do not involve x_t and y_t. In (5), the nominator $(x_k R + y_k u + ((1-k_1)R - u)\triangle_-^*)$ and the denominator $(x_k R + y_k d + ((1-k_1)R - d)\triangle_-^*)$ are wealth levels of the portfolio depending on the state of the market (up or down) after rebalancing the portfolio with $\triangle_-$.

In other words, the PTC sell problem simply balances x and y using $\triangle$ so that the ratio of the up-market portfolio value $(x_k R + y_k u + ((1-k_1)R - u)\triangle_-^*)$ and the down-market portfolio value $(x_k R + y_k d + ((1-k_1)R - d)\triangle_-^*)$ is w_1. To reflect the role of w_1 and w_2, we will refer to them as *portfolio balancing ratios.*

Now, suppose that we have a pair (x_i, y_i) whose optimal $\triangle_+$ (or $\triangle_-$) is 0. Then, (x_i, y_i) is a pair that does not require rebalancing. From the defintion of (5) and (6) (x, y) pairs whose optimal $\triangle$ is 0 are on the following lines

$$w_1 = \frac{x_k R + y_k u}{x_k R + y_k d}, \quad w_2 = \frac{x_k R + y_k u}{x_k R + y_k d}. \tag{8}$$

Let us call the equations in (8) as (8.1)(left) and (8.2)(right).
Rearranging (8.1) or (8.2), we get

$$y = \frac{R(w_1 - 1)}{u - w_1 d} x, \quad y = \frac{R(w_2 - 1)}{u - w_2 d} x. \tag{9}$$

Let us refer to these lines as (9.1) and (9.2). Since we assume that (x, y) stays in $\mathcal{S}$, (9.1) and (9.2) make sense only when

$$0 \le w_1 \le u/d, \quad 0 \le w_2 \le u/d.$$

Note that two equations in (8) form two distinct lines in terms of (x_k, y_k) given w_1 and w_2.

Lemma 4.1. *If a given portfolio (x_k, y_k) satisfies (8.1) or (8.2), it does not need rebalancing.*

Proof. If (x_k, y_k) satisfies (8.1), $\triangle_- = 0$ and if (x_k, y_k) satisfies (8.2), $\triangle_+ = 0$. Therefore, (x_k, y_k) does not require rebalancing in either cases. □

Before we proceed, let us introduce a property of w_1 and w_2.

Lemma 4.2. *Suppose that $k = k_1 = k_2$. Then $w_1 \geq w_2$.*

Proof. Since $x^{1/\gamma}$ is a monotonically increasing function, we have $x^{1/\gamma} \geq y^{1/\gamma}$ if and only if $x \geq y$. Therefore $w_1 \geq w_2$ if and only if

$$-\frac{\pi_u((1-k_1)R-u)}{\pi_d((1-k_1)R-d)} \geq -\frac{\pi_u((-1+k_2)R+u)}{\pi_d(-(1+k_2)R+d)}. \tag{10}$$

(10) is true because it can be reduced to $d \leq u$. Hence, $w_1 \geq w_2$. □

From (9.1) and (9.2), we can see that they meet at $(x, y) = (0, 0)$. When combine this fact with lemma 4.2, we can see that (9.1) and is above the line (9.2) in $\mathcal{S}$. ($R(w_1-1)/(u-w_1d) \geq R(w_2-1)/(u-w_2d)$ when $w_1 \geq w_2$.) Based on arguments so far, we can see that (9.1) and (9.2) separates $\mathcal{S}$ into 3 parts; sell-region, NTR, and buy-region.

Let us define $\triangle_-$ that satisfies (5) as $\triangle_-^*$ and $\triangle_+$ that satisfies (6) as $\triangle_+^*$, respectively. The following lemma is used to show that the sign of $\triangle_-^*$ changes depending on the location of an (x, y) in $\mathcal{S}$.

Lemma 4.3.

$\triangle_-^* \geq 0,$ *when (x, y) is on the upper left of (9.1)*
$\triangle_-^* = 0,$ *otherwise.*

$\triangle_+^* \geq 0,$ *when (x, y) is on the lower right of (9.2)*
$\triangle_0^* = 0,$ *otherwise.*

Proof. We solve $\triangle_-^*$'s case only. $\triangle_+^*$'s case is a mirror image. Suppose that (x_1, y_1) is on the upper left of (9.1). Then, (x_1, y_1) can be written as $(x_0, y_0 + \alpha)$ where $\alpha \geq 0$ and (x_0, y_0) satisfies (9.1). From (5) we have

$$w_1(x_0R + y_0d + ((1-k_1)R-d)\triangle_-^*) + \alpha w_1 d = x_0R + y_0u - KR((1-k_1)R-u)\triangle_-^*) + \alpha u.$$

Since (x_0, y_0) satisfies (9.1), we have

$$w_1(x_0R + y_0d) = x_0R + y_0u.$$

Subtracting these two equations, we get

$$w_1((1-k_1)R-d)\triangle_-^* + \alpha w_1 d = ((1-k_1)R-u)\triangle_-^* + \alpha u.$$

Rearranging the equation above, we get

$$\triangle_-^* = \frac{\alpha(u - w_1d)}{w_1((1-k_1)R-d) - ((1-k_1)R-u)} \geq 0.$$

Therefore, $\triangle_-^* \geq 0,$ when (x, y) is on the upper left of (9.1). The rest of the lemma can be proven using similar arguments. □

Lemma 4.3 indicates that $\triangle_-^* > 0$ and $\triangle_+^* > 0$ cannot happen simultaneously.

Lemma 4.4. *$g_-(\triangle_-)$ and $g_+(\triangle_+)$ are concave functions with respect to $\triangle_-$ and $\triangle_+$ within $\mathcal{S}$.*

Proof.

$$\begin{aligned} g_-''(\triangle_-) :=\ & -\gamma\pi_u((1-k_1)R-u)^2(x_kR+y_ku+((1-k_1)R-u)\triangle_-)^{-\gamma-1} \\ & -\gamma\pi_d((1-k_1)R-d)^2(x_kR+y_kd+((1-k_1)R-d)\triangle_-)^{\gamma-1} \\ g_+''(\triangle_+) :=\ & -\gamma\pi_u(-(1+k_2)R+u)^2(x_kR+y_ku+(-(1+k_2)R+u)\triangle_+)^{-\gamma-1} \\ & -\gamma\pi_d(-(1+k_2)R+d)^2(x_kR+y_kd+(-(1+k_2)R+d)\triangle_+)^{\gamma-1} \end{aligned}$$

Second derivatives of $g_-(\triangle_-)$ and $g_+(\triangle_+)$ are negative within $\mathcal{S}$. Therefore, $g_-(\triangle_-)$ and $g_+(\triangle_+)$ are concave functions. □

Since $g_-(\triangle)$ and $g_+(\triangle)$ are concave functions, they have unique optimums at $\triangle_-^*$ and $\triangle_+^*$, respectively. Furthermore, when $\triangle_- = 0$ and $\triangle_+ = 0$, we have $g_-(\triangle_-) - g_-(0) = 0$ and $g_+(\triangle_+) - g_-(0) = 0$. Putting these facts together, we have corollary 4.5.

Corollary 4.5. *Let us define functions $\theta_-(\triangle_-) := g_-(\triangle_-) - g_-(0)$ and $\theta_+(\triangle_+) := g_+(\triangle_+) - g_+(0)$. Then $\theta_-(\triangle_-)$ and $\theta_+(\triangle_+)$ are concave within $\mathcal{S}$ and they go through the origin.*

Proof. It follows naturally from lemma 4.4 and

$$\theta_-(0) := g_-(0) - g_-(0) = 0, \quad \theta_+(0) := g_+(0) - g_+(0) = 0.$$

□

Now, let us define NTR of the PTC sell-problem and buy-problems as follow.

$$\begin{aligned} \Theta_-^N(x_k, y_k) &:= \{\triangle_- : \theta_-(\triangle_-) < 0, \triangle_- \le \triangle_-^*\}, \\ \Theta_+^N(x_k, y_k) &:= \{\triangle_+ : \theta_+(\triangle_+) < 0, \triangle_+ \le \triangle_+^*\}. \end{aligned}$$

Corollary 4.6. *$\theta_-(\triangle_-)$ and $\theta_+(\triangle_+)$ are concave functions when $\triangle_-$ and $\triangle_+$ are within $\mathcal{S}$.*

Proof. $\theta_-(\triangle_-)$ $(\theta_+(\triangle_+))$ is different from $g_-(\triangle_-)$ $(g_+(\triangle_+))$ by a constant. Proof is the same as that of lemma 4.4. □

Based on observations so far, we have the following lemma.

Lemma 4.7. *When $\triangle_-^* > 0$ $(\triangle_+^* > 0)$, $\Theta_-^N(x, y) \equiv \emptyset$ $(\Theta_+^N(x, y) \equiv \emptyset)$.*

Proof. $\theta_-(0) = 0$ and $\theta_+(0) = 0$. $\theta_-(\triangle_-) = 0$ $(\theta_+(\triangle_+ = 0))$ achieves a unique optimum at $\triangle_- = \triangle_-^*$ $(\triangle_+ = \triangle_+^*)$ satisfying $\theta_-(0) \le \theta_-(\triangle_-^*)$ $(\theta_+(0) \le \theta_+(\triangle_+^*))$. Since $\theta_-(\triangle_-)$ $(\theta_+(\triangle_+))$ is increasing in the interval $[0, \triangle_-^*]$ $([0, \triangle_+^*])$, $\Theta_-^N(x, y)$ $(\Theta_+^N(x, y))$ is an empty set when $\triangle_-^* > 0$ $(\triangle_+^* > 0)$. □

As you can see in lemma 4.7, $\triangle_-^*$ $(\triangle_+^*)$ does not fall into NTR whenever $\triangle_-^* > 0$ $(\triangle_+^* > 0)$. This lemma indicates that when $\triangle_-^* > 0$ $(\triangle_+^* > 0)$, NTR of the sell (buy) problem is empty. In other words, if we get $\triangle_-^* > 0$ $(\triangle_+^* > 0)$ for a specific pair of (x, y) it is optimal to sell (buy) $\triangle_-^*$ $(\triangle_+^*)$ of a risky asset. Theorem 4.8 summarizes the optimal trading policy for one-period PTC.

Theorem 4.8. *The following is an optimal strategy for* PTC *problem.*

1. *Given* (x, y)*, calculate* $\triangle_-^*$ *and* $\triangle_+^*$*.*

2. *If* $\triangle_-^* > 0$*, liquidate* $\triangle_-^*$ *of risky asset.*

3. *If* $\triangle_+^* > 0$*, purchase* $\triangle_+^*$ *of risky asset.*

4. *Otherwise, do nothing.*

Proof. Based on lemma 4.3, only 3 mutually exclusive cases can happen.

Case 1: $\triangle_-^* > 0$. (Sell)

Case 2: $\triangle_+^* > 0$. (Buy)

Case 3: $\triangle_-^* = \triangle_+^* = 0$. (Do nothing.)

When $\triangle_-^* > 0$ ($\triangle_+^* > 0$) happens, selling $\triangle_-^* > 0$ (buying $\triangle_+^*$) is optimal because $\Theta_-^N(x_k, y_k)$ ($\Theta_+^N(x_k, y_k)$)is empty and PTC sell-problem (buy-problem) is concave. Therefore, taking above steps is an optimal strategy. □

In subsection 4.2 we construct an optimal strategy for FPTC using techniques that are used in this section.

4.2 Analysis of FPTC

As we did in subsection 4.1, we can easily show that (2) and (3) are concave functions.

Lemma 4.9. $f_-(\triangle_-; x_k, y_k)$ *and* $f_+(\triangle_+; x_k, y_k)$ *are concave functions in terms of* $\triangle_-$ *and* $\triangle_+$*.*

Proof. Similar to the proof of lemma 4.4. □

For simplicity, sometimes we will drop subscripts for x_k, y_k. Note that $f_0(0)$ needs to be specified separately in FPTC's case because there is a jump in the utility function when $\triangle = 0$. Introduction of the fixed transaction cost to the portfolio optimization causes certain changes. Note that most analyses in section is about the case of $\mid \triangle \mid > 0$ because the case of $\mid \triangle \mid = 0$ is simple. First of all, two lines, (5) and (6), have intercept terms as below.

$$w_1 = \frac{x_k R + y_k u - KR + ((1 - k_1)R - u)\triangle_-^*}{x_k R + y_k d - KR + ((1 - k_1)R - d)\triangle_-^*} \tag{11}$$

and

$$w_2 = \frac{x_k R + y_k u - KR + (-(1 + k_2)R + u)\triangle_+^*}{x_k R + y_k d - KR + (-(1 + k_2)R + d)\triangle_+^*}, \tag{12}$$

where w_1 and w_2 are defined as (7).

$$\triangle_-^* = \frac{w_1(x_k R + y_k d - KR) - (x_k R + y_k u - KR)}{-w_1((1 - k_1)R - d) + ((1 - k_1)R - u)} \tag{13}$$

and

$$\triangle_+^* = \frac{w_2(x_kR + y_kd - KR) - (x_kR + y_ku - KR)}{-w_2(-(1+k_2)R + d) + (-(1+k_2)R + u)}. \tag{14}$$

Like (9) we define lines that separates $\mathcal{S}$ into three regions as

$$w_1 = \frac{x_kR + y_ku - KR}{x_kR + y_kd - KR}, \quad w_2 = \frac{x_kR + y_ku - KR}{x_kR + y_kd - KR}, \tag{15}$$

where w_1 and w_2 are the same as those of PTC. These lines can rewritten as below.

$$y = \frac{R(w_1 - 1)}{u - w_1d}x - \frac{KR(w_1 - 1)}{u - w_1d}, \quad y = \frac{R(w_2 - 1)}{u - w_2d}x - \frac{KR(w_2 - 1)}{u - w_2d}. \tag{16}$$

One main difference between (9) and (16) is the existence of intercepts. Unlike (9) , lines (16) usually do not go through the origin. Two lines in (16), meet at $(x = K, y = 0)$. Since (16) uses the same w_1 and w_2, we still have $w_1 \geq w_2$ and (16.1) lies above (16.2) in $\mathcal{S}$. Besides these changes, most analyses in subsection 4.1 hold.

Lemma 4.10.

$\triangle_-^* \geq 0,$ *when* (x, y) *is on the upper left of (16.1)*
$\triangle_-^* = 0,$ *otherwise.*

$\triangle_+^* \geq 0,$ *when* (x, y) *is on the lower right of (16.2)*
$\triangle_0^* = 0,$ *otherwise.*

Proof. This lemma can be proved using lemma 4.3. □

Like PTC's case, lemma 4.10 indicates that lines (16.1) and (16.2) into sell-region, buy-region and NTR. Explicit formulae for $\triangle_-^*$ and $\triangle_+^*$ in FPTC's case can be written as

$$\triangle_-^* = \frac{-R(w_1 - 1)x + (u - w_1d)y + (w_1 - 1)KR}{w_1((1 - k_1)R - d) - ((1 - k_1)R - u)} \tag{17}$$

and

$$\triangle_+^* = \frac{-R(w_2 - 1)x + (u - w_2d)y + (w_2 - 1)KR}{w_2(-(1 + k_1)R + d) - (-(1 + k_1)R + u)}. \tag{18}$$

Let us define the following functions, which are similar to $\theta_-(\triangle_-)$ and $\theta_+(\triangle_+)$ in PTC's case.

$$\delta_-(\triangle_-) := f_-(\triangle_-) - f_0(0), \quad \delta_+(\triangle_+) := f_+(\triangle_+) - f_0(0).$$

Let us also define NTR of FPTC as below. In the remainder of this section, we will show that $\Theta_-(x, y)$ ($\Theta_+(x, y)$) can be non-empty even when $\triangle_-^* > 0$ ($\triangle_-^* > 0$); and analyze the shape of $\Theta_-(x, y)$ ($\Theta_+(x, y)$). Non-emptiness of $\Theta_-(x, y)$ and $\Theta_+(x, y)$ means that it is possible that we may not take $\triangle_-^*$ or $\triangle_-^*$ even if they are positive.

As we mentioned in subsection 4.1, $\theta_-(\triangle_-)$ and $\theta_+(\triangle_+)$ go through the origin because PTC does not consider the fixed transaction cost. However, $\delta_-(\triangle_-)$ ($\delta_+(\triangle_+)$) has a negative y-intercept because there is a jump at $\triangle_- = 0$ ($\triangle_+ = 0$), where the fixed transaction cost disappears.

In subsection 4.1, we defined NTR in terms of $\triangle$ and learned that taking $\triangle_-^* > 0$ ($\triangle_+^* > 0$) is an optimal strategy for the PTC sell-problem (buy-problem). In this section, we define

NTR slightly differently. Instead of describing NTR in terms of $\triangle$, we will describe it in terms of $\triangle_-^*$ and $\triangle_+^*$. In subsection 4.1, it was convenient to describe NTR in terms of $\triangle$ to show that taking any positive $\triangle_-^*$ (or $\triangle_+^*$) is an optimal strategy. However, in this section, we describe NTR in terms of $\triangle_-^*$ and $\triangle_+^*$ to show that even the positive FOC(first order optimality condition) solution may not be taken due to the existence of the fixed transaction cost. Since a fixed $(x_k,\ y_k)$ pair completely specifies $\triangle_-^*$ and $\triangle_+^*$, our analysis specifies NTR in terms of $(x_k,\ y_k)$.
Furthermore, we also define NTR as below

$$\begin{aligned} \Theta_- &:= \{(x,y) : \delta_-(\triangle_-^*(x,y)) \leq 0\}, \\ \Theta_+ &:= \{(x,y) : \delta_+(\triangle_+^*(x,y)) \leq 0\}. \end{aligned} \tag{19}$$

Note that we used equality in (19) because there is no transaction even if $\delta_-(\triangle_-^*(x,y)) = 0$ or $\delta_+(\triangle_+^*(x,y)) = 0$. We can check if a given (x,y) belongs to the no-trading region using theorem 4.11.

Theorem 4.11. *One can check if a given point (x,y) belongs to Θ_- or Θ_+ using the following.*

$$\begin{aligned} \Theta_- \ &:= \{(x,y) : (\pi_u w_1^{1-\gamma} + \pi_d) u(xR + yd - KR + ((1-k_1)R - d) \triangle_-^* (x,y)) \\ &\leq \pi_u u(xR + yd) + \pi_d u(xR + yu)\}. \end{aligned} \tag{20}$$

$$\begin{aligned} \Theta_+ \ &:= \{(x,y) : (\pi_u w_2^{1-\gamma} + \pi_d) u(xR + yd - KR + (-(1+k_2)R - d) \triangle_+^* (x,y)) \\ &\leq \pi_u u(xR + yd) + \pi_d u(xR + yu)\}. \end{aligned} \tag{21}$$

Proof. For simplicity, we will only discuss the FPTC sell-problem. Optimal policy for one-period FPTC is simply taking $\triangle_-^* > 0$ if selling $\triangle_-^*$ is better than not trading at all. Therefore, we can specify the no-trading region by stating Θ_- in-terms of (x,y). Since $\triangle_-^*$ satisfies (11), we can write $f_-(\triangle_-^*; x, y)$ as follows.

$$\begin{aligned} f_-(\triangle_-^*; x, y) \ &:= \pi_u u(xR + yu - \tau_- + ((1-k_1)R - u)\triangle_-^*) \\ &+\pi_d u(xR + yd - KR + ((1-k_1)R - d)\triangle_-^*) \\ &= \pi_u \left(u((w_1)(xR + yd - KR + ((1-k_1)R - d)\triangle_-^*)) \right) \\ &+\pi_d u(xR + yd - KR + ((1-k_1)R - d)\triangle_-^*) \\ &= (\pi_u w_1^{1-\gamma} + \pi_d) u(xR + yd - KR + ((1-k_1)R - d)\triangle_-^*) \end{aligned} \tag{22}$$

Since $\triangle_-^*$ is a function of (x,y), (20) specifies Θ_- in terms of (x,y) only. Θ_+'s case is omitted because it is a mirror-image of Θ_-'s case. $\square$

Theorem 4.11 completely describes NTR because it easy to check if a certain (x,y) pair falls into the no-trading region using it. Figure 4.2 depicts the no-trading region of FPTC for the following parameters.

$$\begin{gathered} \pi_u = 0.5, \quad \pi_d = 0.5, \quad u = 1.3, \quad d = 0.9, \quad R = 1.04, \\ k_1 = 0.003, \quad k_2 = 0.003, \quad \gamma = 2, \quad K = 0.13. \end{gathered}$$

Theorem 4.12. *The following is an optimal strategy for FPTC problem.*

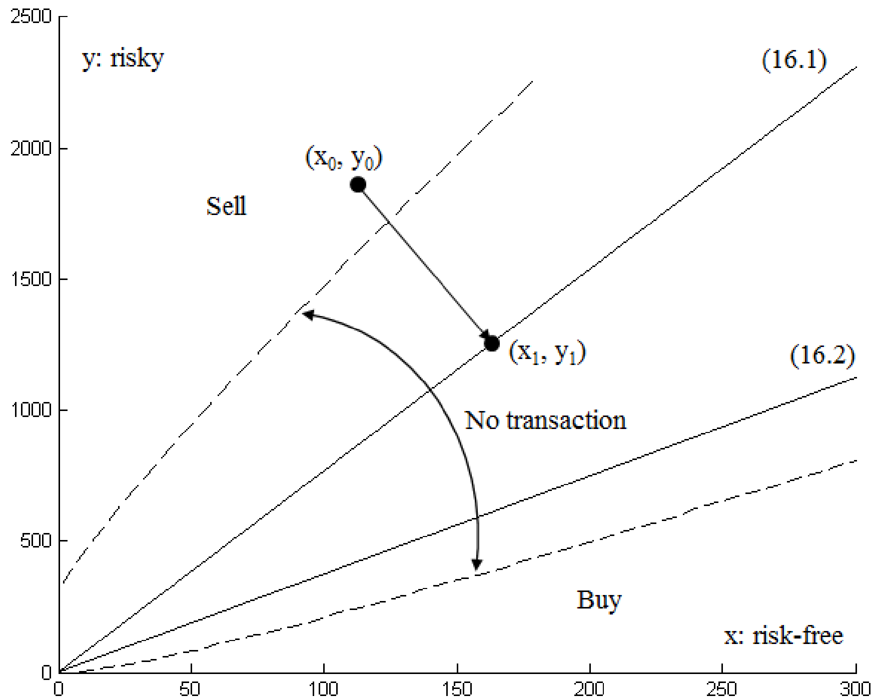

Figure 1: NTR of FPTC

1. *Given (x, y), calculate $\triangle_-^*$ and $\triangle_+^*$.*

2. *If $\triangle_-^* > 0$ and $(x, y) \notin \Theta_-$, liquidate $\triangle_-^*$ of risky asset.*

3. *If $\triangle_+^* > 0$ and $(x, y) \notin \Theta_+$ purchase $\triangle_+^*$ of risky asset.*

4. *Otherwise, do nothing.*

Proof. Based on lemma 4.10, only 3 mutually exclusive cases can happen.

Case 1: $\triangle_-^* > 0$.

Case 2: $\triangle_+^* > 0$.

Case 3: $\triangle_-^* = \triangle_+^* = 0$.

When $\triangle_-^* > 0$ ($\triangle_+^* > 0$) it is an optimal solution for the FPTC sell-(buy)problem because f_- (f_+) is a concave function. We accept $\triangle_-^*$ or $\triangle_+^*$ as an optimal solution only if it is better than $f_0(\triangle = 0)$. (Application of the branch-and-bound algorithm.) Liquidating (Purchasing) $\triangle_-^*$ ($\triangle_+^*$) of risky a asset is better than not trading only if $(x, y) \notin \Theta_-$ ($(x, y) \notin \Theta_+$) from theorem 4.11. Therefore, taking steps above is an optimal strategy. □

In the remainder of this section, we analyze the shape of Θ_- and Θ_+ in terms of x and y, which turn out to be indefinite in most cases. To proceed, let us define $q(x, y) := (ax + by + c)^{\gamma-1}/(\gamma - 1)$. Then its hessian is

$$\bigtriangledown^2 q(x,y) := -\gamma(ax+by+c)^{-\gamma-1}\begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}. \tag{23}$$

Note that (23) is a rank-one negative definite matrix. Using (23), we can write the Hessian of $\delta_-(\triangle^*_-(x,y))$ as below. (Buy-problem's case is omitted because it is a mirro-image of sell-problem's case.)

$$\begin{array}{rl} \bigtriangledown^2\Theta_- = & -\gamma\pi_u(xR+yu-KR+((1-k_1)R-u)\triangle^*_-)^{-\gamma-1}\times A \\ & -\gamma\pi_d(xR+yd-KR+((1-k_1)R-d)\triangle^*_-)^{-\gamma-1}\times B \\ & +\gamma\pi_u(xR+yu)\triangle^*_-)^{-\gamma-1}\times A' \\ & +\gamma\pi_d(xR+yd)\triangle^*_-)^{-\gamma-1}\times B', \end{array} \tag{24}$$

where

$$A := \begin{bmatrix} R-R(w_1-1)((1-k_1)R-u)/\tau_- \\ u-(uw_1d)((1-k_1)R-u)/\tau_- \end{bmatrix} \begin{bmatrix} R-R(w_1-1)((1-k_1)R-u)/\tau_- \\ u-(uw_1d)((1-k_1)R-u)/\tau_- \end{bmatrix}^T,$$

$$B := \begin{bmatrix} R-R(w_1-1)((1-k_1)R-d)/\tau_- \\ d-(uw_1d)((1-k_1)R-d)/\tau_- \end{bmatrix} \begin{bmatrix} R-R(w_1-1)((1-k_1)R-d)/\tau_- \\ d-(uw_1d)((1-k_1)R-d)/\tau_- \end{bmatrix}^T,$$

$$\tau_- := w_1((1-k_1)R-d)-((1-k_1)R-u),$$

$$A' := \begin{bmatrix} R^2 & Ru \\ uR & u^2 \end{bmatrix} \quad B' := \begin{bmatrix} R^2 & Rd \\ dR & d^2 \end{bmatrix}.$$

In (24), A, A', B, and B' are rank-one positive semidefinite matrices. Since A' and B' have positive signs, $\bigtriangledown^2\Theta_-$ becomes positive semidefinite when $\gamma\pi_u(xR+yu)\triangle^*_-)^{-\gamma-1}$ and $\gamma\pi_d(xR+yd)\triangle^*_-)^{-\gamma-1}$ are significantly larger than $\gamma\pi_u(xR+yu-KR+((1-k_1)R-u)\triangle^*_-)^{-\gamma-1}$ and $\gamma\pi_d(xR+yd-KR+((1-k_1)R-d)\triangle^*_-)^{-\gamma-1}$, which would indicate that Θ_- is a convex space of (x,y). However, according to our tests, Sum of $\gamma\pi_u(xR+yu)\triangle^*_-)^{-\gamma-1}$ and $\gamma\pi_d(xR+yd)\triangle^*_-)^{-\gamma-1}$ is similar to the sum of $\gamma\pi_u(xR+yu-KR+((1-k_1)R-u)\triangle^*_-)^{-\gamma-1}$ and $\gamma\pi_d(xR+yd-KR+((1-k_1)R-d)\triangle^*_-)^{-\gamma-1}$, and $\bigtriangledown^2\Theta_-$ usually turned out to have one positive eigenvalue and one negative eigenvalue. Hence, $\bigtriangledown^2\Theta_-$ is usually indefinite. Like $\bigtriangledown^2\Theta_-$, $\bigtriangledown^2\Theta_+$ is also indefinite in most cases.

So far we showed how to find an optimal solution for a single period problem. In the following section we introduced a heuristic (Algorithm-1) that find an (near) optimal solution for a multi-period FPTC problem and report computational results.

5 Numerical implications

Computational test results in this section is presented in subsections 5.1∼5.3. A heuristic that can produce an approximate solution for multi-period FPTC is introduced in subsection 5.1. subsection 5.2, depicts how changing each parameter affects NTR. In subsection 5.3, we compare our results to actual asset allocations of several world-renowned PSPs.

5.1 A heuristic for multi-period FPTC

In this subsection we introduce a heuristic for solving multi-period FPTC problems. For testing purpose, we used the following benchmark parameters: $r_b = 1.0371$, $r_s = 1.0771$, $\sigma_s = 0.2$, $\gamma = 2$, $k_1 = 0.005$, $k_2 = 0.005$, K is 0.1% of the total wealth. $r_b = 1.0371$ was taken from Bodie, Markus, and Kane (2005) and r_s is set to 1.0771 so that risk-premium is around 4%. From these parameters, we can compute $\pi_d = 0.5$, $\pi_u = 0.5$, $d = 0.8771$, and $u = 1.2771$. Unless stated otherwise, FPTC problems in this paper will use these parameters.

From now on we let ϕ denotes the percentage change in net contribution. In this subsection $\phi \in \{-5\%, 0\%, +5\%\}$. Let us give a simple example. Suppose that $\phi = -5\%$. Then, x is decreased by $0.05 \times (x + y)$ at the beginning of each period for T periods.

Taking net-contribution into account for optimal asset allocation is worthwhile for the following reason. First, most PSPs have quite accurate estimates on expected ϕ of the near future. Simply put, a PSP manager can easily come by future net-contribution values. Second, as we will show later in subsection 5.2, ϕ has a significant impact on optimal asset allocations. To check how changes of ϕ affect FPTC's optimal allocation, Algorithm-1 was implemented so that it can take periodic changes in x by net-contribution into account.

The algorithm we introduce in this section can solve FPTC approximately where T is relatively large. However, since the running time of our brute-force method increases exponentially as the number of periods increases, it may be difficult to run this algorithm for more than several (up to 5 or 6) periods with a relatively small ϵ. Algorithm-1 in the appendix produces an approximate solution for a discrete-time multi-period portfolio optimization problem. We can expect to get a near-optimal solution if a small ϵ is used.

In Algorithm-1, input arguments k_1, k_2, K,d, u, π_d, π_u, r_b are constant values, ϕ is an annual changes in total asset, and $\epsilon > 0$ is an offset. In Algorithm-1, 'Loop $\delta = -y : \epsilon : x$' on $[L06]$ repeats $[L07] \sim [L14]$ with different values of δ, which goes from $-y$ to x with a step size ϵ.

Certainly, we can solve the whole problem using a brute force method without $[L17] \sim [L18]$. However, having $[L17] \sim [L18]$ in the algorithm reduces the running time of the algorithm significantly because they provide us with 1-period optimal solutions at 2^{T-1} nodes at time $T - 1$. Put otherwise, if we try n possible values at these nodes, it would require $2^{T-1} \cdot n$ trials. But since we can find an optimal choice at each node of time $T - 1$, its running time for time $T - 1$ would be 2^{T-1} instead.

We ran this algorithm at various points within a grid $(x, y) \in [0, 4,000] \times [0, 4,000]$ for different values of ϕ in $\{-5\%, 0\%, +5\%\}$ to see how 3-period optimal trading policy changes depending on ϕ. (Throughout this section, we report test results on 3-period FPTC problems.) Figure 2 depicts no-trading region (between dotted lines) and rebalancing lines (solid lines) of the 3-period benchmark problem using above-mentioned parameters where $\phi = 0$. Figure 2 is constructed by solving 3-period FPTC at various points within $[0, 4,000] \times [0, 4,000]$. When an initial (x, y) pair lies outside of the no-trading region, it is optimal to trade so that resulting asset allocation falls onto the cloest rebalancing lines. Otherwise, not making transaction is the optimal trading strategy. Note that NTR is bounded above and below by two curves, not straight lines, that make NTR an indefinite space.

Figure 3 shows NTR of $1 \sim 3$-period FPTC problems. According to Figure 3, NTR shrinks as we increase the number of periods in FPTC. There is a big difference between the sizes of

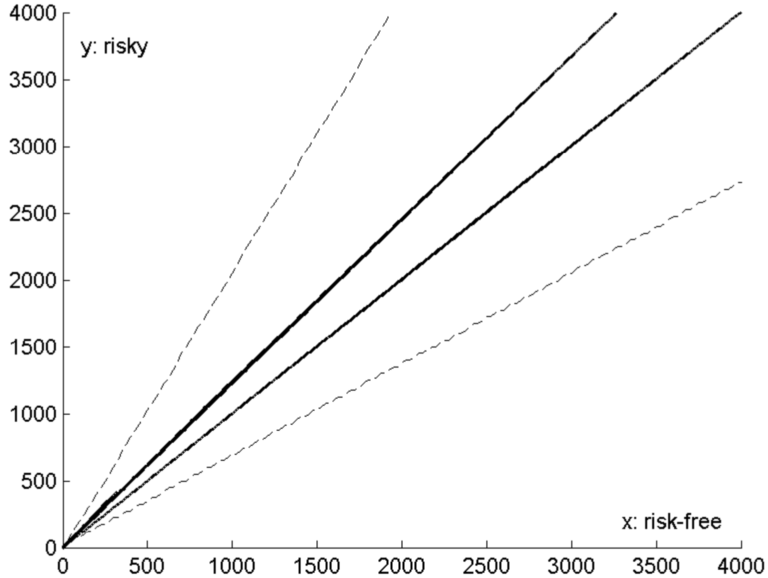

Figure 2: Benchmark 3-period optimal trading policy $\phi = 0$

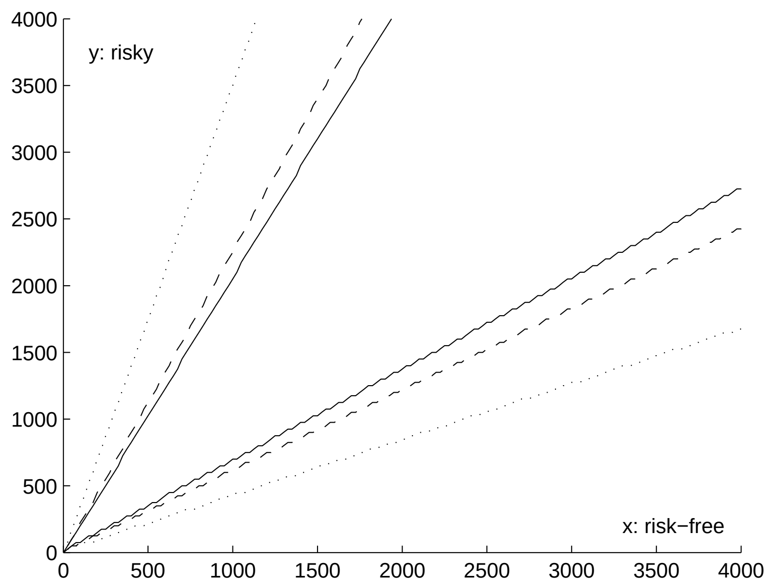

Figure 3: NTR of $1 \sim 3$ period optimal solutions: solid (3-period), dashed (2-period), dotted (1-period)

1-period FPTC and 2-period FPTC NTR and the reduction in the size of NTR gets smaller as the number of periods decreases. This change in the size of NTR can be explained as follows. Suppose that a (x_0, y_0) pair falls on the upper (sell-region and NTR boundary) boundary of 1-period FPTC problem's NTR meaning that profit from trading is almost equivalent to the loss from trading cost. Now, suppose that we would like solve a 2-period FPTC problem with the same initial assets (x_0, y_0). Since the expected return from stock increase in each period is positive, overall profit from trading would be greater than trading cost hence allowing one to trade for the 2-period FPTC problem. For this reason, NTR shrinks as the number of periods, T, increases.

In this subsection, we introduced a heuristic for solving a multi-period FPTC. In the following subsection, we explain how parameter changes affect NTR of 3-period FPTC.

5.2 Sensitivity analysis

In this section, we explina how changing each of 6 parmeters (ϕ, K, (k_1, k_2), γ, r_s, and σ) affects NTR of 3-period FPTC. This is done by solving 3-period FPTCat various combinations of $(x, y) \in [0, 4,000] \times [0, 4,000]$ by changing each parameter while holding the others fixed. Test result of this subsection can be summarized by Table 3.

Table 3: NTR change in response to a parameter *increase*

Parameter	Par. notation	Par. values	Figure #	NTR change
Net-balance	ϕ	$-5\%, 0\%, +5\%$	4	Pushed up
Fixed cost	K	$0.1\%, 0.5\%, 1\%$	5	Widened
Prop. cost	k_1, k_2	$0.1\%, 0.5\%, 1\%$	6	Widened
Risk aversion	γ	$1.9, 2.0, 2.1$	7	Pushed down
Risky asset return	r_s	$1.0771, 1.0810, 1.0732$	8	Pushed up
Volatility	σ	$1.9, 2.0, 2.1$	9	Pushed down

Table 3 has 5 columns: Parameter, Par. Notation, Par. values, Fgure # and NTR change. 'Parameter' tells us which parameter's sensitivity was tested. 'Par. Notation' is shows the notation for each parameter. 'Par. values' are tried values for each parameter. 'Figure #' directs us to a figure that contains sensitivity test for each parameter. Finally, 'NTR change' tells how NTR changes as a parameter increases. As you can see in Table 3, most parameter sensitivity test results are intuitive.

As ϕ increases NTR is pushed up. When ϕ is a positive number, it is expected that a PSP would receive a lump sum of cash addition to x at the beginning of each year. In other words, even if a PSP has more wealth on y, x to y ratio would be optimally balanced by receiving more cash on x. On the other hand, when ϕ is a negative number, it is expected that a PSP will lose part of x at the beginning of each year hence requiring more allocation of its wealth on x.

In Figure 5, we can see that NTR widens as K (fixed trading cost as a portion of total asset value) increases. Let us give a simple example. Suppose that K is 0.1% of the total value of assets and maximum expected profit from trading is 0.5% the total value of assets. Then, it is better to trade. However, if we increase K to 1%, it is no more an optimal to trade. Like this small example, as we increase K NTR would be widened too. As you can see in Figure 5, changes in K seems to have a big effect on the size of NTR. When K is about 1% of the total asset, NTR covers most of the positive quadrant.

Like K, larger k_1 and k_2 widen NTR for the same reason as the case of K. However, the impact of changes in k_1 and k_2 is smaller than K possibly because k_1 and k_2 are paid proportional to the risky asset only.

In addition, we tried 3 values for γ, which were 1.9, 2.0 and 2.1. As Figure 7 shows smaller γ makes an investor less sensitive to the risk and force him/her to invest more money on a risky asset lifting the NTR. This is because a larger γ means a higher order for the CRRA utility function, which becomes more sensitive to changes.

In Figure 8, larger values of r_s let an investor to invest more on a risky asset because higher expected return from a risky asset makes investing on a risky asset more attractive.

Figure 9 depicts how NTR change for different values of σ. Like the case of γ, smaller values of σ lift NTR because smaller volatility in a risky asset means less risk on a risky asset hence making a risky asset a more attractive investment opportunity.

In this subsection, we investigated how changing each parameter affects NTR. In the following subsection, we apply our asset allocation scheme to 5 large-scale PSPs and compare theoretical asset allocations to their actual asset allocations.

5.3 Application to pension funds

In subsection 5.2, we studied how 3-period optimal rebalancing strategies changes depending on 6 input parameters. In this subsection, we apply our optimal asset allocation scheme to some world-renowned pension funds (whose detailed financial information is divulged to public) and compare our theoretical results to their actual asset allocations.

Tables 12∼16 include tests results on 5 PSPs that we studied. Each of these tables is made of two parts. Top halves of them are tests results without applying ϕ and the bottom halves that reflect ϕ. In each row of Table 12∼16, we solved 9 problems using actual x(risk-free asset value) and y(risky asset value) of each PSPwith parameter combinations $\sigma \in \{10\%, 15\%, 20\%\} \times (k_1, k_2) \in \{0.5\%, 1.5\%, 2.5\%\}$ while holding the other parameters fixed. For each problem in Table12∼16, we applied the same parameters for 3-periods (or years).

To point a certain spot in these tables, let us use [PSP name, up(ϕ not applied) or down(ϕ applied), Year, Vol.= #, k_1, k_2 = #]. For example, [NPS, up, 2004, Vol.= 10%, $k_1, k_2 = 1.5\%$] points to the upper part of Table 16 for year 2004, where Volatility is 10% and k_1, k_2 are 0.15%, which is 81.17%. Likewise, [CALP, down, 2006, Vol.= 15%, $k_1, k_2 = 0.5\%$] is 85.06%. In addition, we use $*$ to denote a wildcard. For example [CALP, up, *, Vol.= 15%, $k_1, k_2 = 0.5\%$] indicates risky asset allocations of CALP between 2002 $\sim$ 2011, where Vol.= 15% and $k_1, k_2 = 0.5\%$.

Since Tables 12∼16 lots of information, we summarized data in them in Tables 4, 5, and 6, which show us impacts of stock market volatility, liquidity (or proportional trading cost), and net contribution, respectively, on theoretical optimal asset allocations.

First, Table 4 contains average risky asset allocations of each PSP for different volatilities (or σ), where ϕ is not applied. Columns 2, 4, 6 are average values of optimal asset allocations in upper part of Tables 12∼16 for volatilities $\{10\%, 15\%, 20\%\}$, respectively. For example, 82.06 is an average of [CALP, up, *, Vol.= 10%, $k_1, k_2 = *\%$]. In Table 4, columns 3 and 5 are differences between columns '1 and 3' and '3 and 5'. In this test, we can see that theoretical optimal asset allocation decrease by roughly 10% when market volatility is inceased by 5%. 24.53% drop in risky asset ratio for NPS on column 5 seems to occur because NPS's current asset allocation lies in the buy-region where as other PSPs asset allocations are near the sell-region.

Table 4: Average of optimal asset allocations for different market volatility

	Volatility (σ)				
Name	10%	Difference	15%	Difference	20%
CALP	82.06	-8.23	73.82	-12.06	61.76
NYSLR	83.20	-6.30	76.90	-15.62	61.29
AVON	83.09	-3.99	79.11	-16.99	62.12
FAPF	81.88	-11.31	70.57	-11.96	58.62
NPS	81.99	-12.21	69.78	-24.53	45.26

Table 5 describes how optimal risky asset ratios change as the trading cost increases. Columns 2, 4, 6 are averages of optimal risky asset ratios where trading costs are {0.5%, 1.5%, 2.5%}, respectively, where volatility (σ) is fixed at 20%. This test was conducted so that how liquidity affects optimal risky asset ratio when market crashes. This test indicates that as trading costs increase in 3 fold and 5 fold, optimal risky asset ratio increases by 3% $\sim$ 4% with an exception of FAPF whose asset allocation is near no-trading region and NPS whose asset allocation is in the buy-region.

Table 5: Average of optimal asset allocations for different k_1, k_2 where σ is fixed at 20%

	k_1, k_2				
Name	0.5%	Difference	1.5%	Difference	2.5%
CALP	58.03	3.91	61.94	3.38	65.32
NYSLR	57.90	3.46	61.36	3.25	64.61
AVON	58.05	4.08	62.13	4.04	66.17
FAPF	57.55	1.48	59.03	0.25	59.28
NPS	49.08	-3.64	45.44	-4.19	41.25

It is well known that volatility and bid-ask spread increase when a stock market crash. According to test in Table 4, a myopic investor would try to liquidate around 10% of its risky asset when market volatility increases from 15% to 20%. However, s/he will also see that doing so is not optimal because trading costs (bid-ask spread) jumps too. We can see that this phenomenon is well explained by the test results in Table 4 and Table 5.

Table 6 summarizes how net contribution affects optimal asset allocation. According to test results in Table 6, application of net contribution to FPTC generates $-0.03\% \sim 4.41\%$ changes in optmal asset allocations for 5 PSPs. Since a PSP manages hundreds of millions of dollars, a few percent of a PSP's wealth is a significant amount of money. Knowing that net contribution has a significant impact on optimal asset allocation, we would like to emphasize that it has to be taken into consideration for constructing asset allocation scheme for a PSP.

In this remainder of this subsection, we compare our optimal asset allocations to the actual asset allocations of PSP of our interest. To our astonishment, actual asset allocations

Table 6: Summary of ϕ's impact

			Volatility 10%			Volatility 15%			Volatility 20%				
	ϕ		$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$				
Name	Average	Applied	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	Average	Difference
CALP	-1.1769%	No	92.11	81.16	72.89	85.07	68.71	67.69	58.03	61.94	65.32	72.55	0.55%
		Yes	90.90	80.04	71.90	84.07	69.20	68.49	57.23	61.07	65.05	71.99	
NYSLR	-3.9237%	No	92.09	81.61	75.90	85.05	73.95	71.70	57.90	61.36	64.61	73.80	1.30%
		Yes	88.48	79.95	75.99	82.81	74.17	72.51	55.94	59.58	63.06	72.50	
AVON	0.2324%	No	92.07	81.06	76.14	85.04	76.14	76.14	58.05	62.13	66.17	74.77	4.41%
		Yes	91.57	80.84	73.45	84.74	69.46	64.03	55.26	56.54	57.36	70.36	
FAPF	0.4341%	No	92.15	81.26	72.23	85.11	67.08	59.54	57.55	59.03	59.28	70.36	0.05%
		Yes	92.15	81.17	72.85	85.21	67.59	59.21	57.25	58.35	59.03	70.31	
NPS	0.0076%	No	92.35	80.76	72.87	85.29	67.51	56.55	49.08	45.44	41.25	65.68	-0.03%
		Yes	92.35	80.76	73.00	85.29	67.51	56.55	49.08	45.57	41.25	65.71	

of some PSP matched our test results. Compared our tests, actual allocation of CALP, NYSLR, and FAPF can be well explained by our tests.

For example, [CALP, up, 2002, Vol.= 20%, $k_1, k_2 = 2.5\%$] is 66.69% which is slightly higher than [CALP, down, 2002, Vol.= 25%, $k_1, k_2 = 2.5\%$] = 64.05%, which reflects ϕ. Values of [CALP, up, 2009, Vol.= 20%, $k_1, k_2 = 1.5\%$] and [CALP, down, 2009, Vol.= 20%, $k_1, k_2 = 1.5\%$] are 60.86% and 61.00%, respectively, which are quite close to the actual allocation 60.86%.

According to our tests, NYSLR allocations on a risky asset was lower than our solutions around 2001 and exceeded our optimal asset allocation on a risky asset near 2011. For example, [NYSLR, up, 2001, Vol.= 10%, $k_1, k_2 = 1.5\%$]= 67.05% where its actual stock ratio was 61.60%. In 2011, its stock ratio was 72.04% where our test suggested, [NYSLR, up, 2011, Vol.= 20%, $k_1, k_2 = 0.5\%$]= 56.05%.

Like NYSLR, AVON showed that they allocated more on a risky asset than our schemes. For example, [AVON, up, 2002, Vol.= 20%, $k_1, k_2 = 2.5\%$] = 66.22% whereas AVON actually allocated 79.25%. Overall AVON seems to be allocating more on a risky asset than our solutions.

To our surprise test result on FAPF was quite interesting. In Table 15, we can see that FAPF's actual asset allocations fall in the no-trading region where 'Volatility=20%' and $(k_1, k_2) = 2.5\%$. It seems like FAPF's actual asset allocations can be achieved by solving FPTC with conservative parameter settings such as 'Volatility=20%' and $(k_1, k_2) = 2.5\%$. However, when we look at the lower half of the Table 15, where we took ϕ into account, their asset allocations are different from ours by a few percent. Since, as we have shown in Subsetion 5.2, changes in net contribution have a significant impact on optimal asset allocation, we would recommend a pension fund manager to take ϕ into account for fine tuning their asset allocation policy.

In case of AVON, their risky asset ratio is up to 70 $\sim$ 80%, which is somewhat high when we consider the fact that the world stock market volatility is around 20%. Yet, we would like to emphasize that asset allocation scheme depends on various factors such as risk aversion tendencies (such as γ) and politics.

NPS showed somewhat different asset allocations. First of all, it showed relatively smaller asset allocations on a risky asset investing most of its wealth on a risk-free asset. Since its NPS foundation in 1987, it has been managed in a quite conservative way. However, its steadily increasing stock ratio indicates that it is considering more active participation in stock market in the near future.

6 Conclusion

It has been observed that a group of large pension service providers (PSPs) have maintained stable asset allocation while others have been actively adjusted the mix of their portfolios during the financial turmoil in 2008 and 2009. This observation motivates this paper and it aims to examine an optimal investment strategy of PSPs. In general, although a PSP should ensure stability of investment returns in spite of adverse market conditions as well as maximize returns on investment given risk levels, there are many factors to be taken into account because they affect the investment strategies.

A PSPs asset allocation, this paper constructed a multi-period optimal portfolio while taking fixed and proportional transaction costs into consideration. Also this theoretical portfolio is compared with the actual portfolios of large PSPs. We investigated how no-trading region changes as parameters vary. No-trading region shrinks as the number of period decreases. Increasing fixed and proportional trading costs widen no-trading region because larger trading costs will make an investor less willing to make a positive amount of trade. The change in the size of no-trading region is more sensitive to the fixed cost changes. Decreasing stock market volatility and risk aversion parameter lift the no-trading region allowing an investor to allocate more wealth to a risky asset.

An interesting finding in this paper is the fact that the change in net-contribution has a significant impact on the optimal asset allocation. Unlike other unpredictable factors, most PSP's have pretty accurate estimates for the future expected changes in net-contribution. Therefore, we strongly advise a PSP to take the future expected changes in net-contribution into account to come up with an optimal asset allocation.

A bid-ask spread and a stock market volatility are indicators of a stock market condition. When there is a stock market crash, a myopic optimal choice is to liquidate as much stocks as possible. Yet an increase in a bid-ask spread, which occurs often together with a stock market crash, increases the trading costs preventing an investor from liquidating more. This liquidity problem is considered in the optimal portfolio strategies.

Since we conducted our tests with certain limitations without reflecting each PSP's environment in detail, our optimal allocation can be far from an actual optimal asset allocation for each PSP. Yet, in this paper we discovered that some PSP's actual asset allocations can be well explained by our test results. For example, actual asset allocations of CALP, NYSLR and FAPF could be well explained by our test results. AVON seems to be investing around 10% more on a risky asset than our optimal asset allocation schemes. Our test shows that the pension fund management strategies of some PSP's are close to the theoretical optimal allocations, but some have room to improve. NPS's asset allocation on a risk-free asset seems to be too high, but it has been steadily increasing the ratio of a risky asset, which is a recommendable movement by our optimal strategies.

Regarding the limitation of our current research approach, we suggest to consider the followings for future research. In this paper we only considered two types of assets, risk free and risky assets as the guide line of top-down approach. More realistic and various choices of asset classes could be considered for optimal investment strategies. Tax would impact the strategies. More detailed investigation can be conducted regarding the liquidity problem including price impact.

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7 Appendix

Table 7: CALP: asset allocation and cash flow

Year	Stock	Bond	Stock ratio	Cont.	Benefit	NC	NC ratio
2002	80,220,734	40,064,811	66.69%	2,955,706	6,534,405	-3,578,699	-2.97%
2003	86,135,240	37,904,304	69.44%	3,812,969	7,105,939	-3,292,970	-2.65%
2004	102,505,858	42,991,871	70.45%	6,527,792	7,790,611	-1,262,819	-0.86%
2005	114,838,218	54,340,778	67.88%	8,950,901	8,549,355	401,546	0.23%
2006	129,887,184	52,026,254	71.4%	9,175,908	9,407,002	-231,094	-0.12%
2007	149,704,501	61,218,906	70.98%	9,705,083	10,252,129	-547,046	-0.25%
2008	122,375,605	59,934,932	67.12%	10,754,877	11,066,832	-311,955	-0.17%
2009	80,229,982	51,597,650	60.86%	10,794,731	12,018,619	-1,223,888	-0.92%
2010	91,941,405	53,359,863	63.28%	10,333,916	13,154,844	-2,820,928	-1.94%
2011	116,731,425	53,066,388	68.75%	11,065,486	14,600,037	-3,534,551	-2.08%

Table 8: NYSLR: asset allocation and cash flow

Year	Stock	Bond	Stock ratio	Cont.	Benefit	NC	NC ratio
2003	51,357,030	32,019,681	61.60%	980,853	5,029,766	-4,048,913	-4.86%
2004	74,876,438	29,691,227	71.61%	1,585,474	5,423,277	-3,837,803	-3.67%
2005	80,917,186	20,100,078	80.10%	3,314,919	5,690,865	-2,375,946	-2.35%
2006	88,550,861	19,868,995	81.67%	3,117,876	6,072,868	-2,954,992	-2.73%
2007	90,119,680	33,536,212	72.88%	3,100,572	6,431,731	-3,331,159	-2.69%
2008	58,582,079	10,500,845	84.80%	3,030,236	6,883,034	-3,852,798	-5.58%
2009	47,870,996	36,541,603	56.71%	2,885,457	7,265,499	-4,380,042	-5.19%
2010	72,673,981	33,726,066	68.30%	2,719,494	7,718,872	-4,999,378	-4.70%
2011	79,953,953	31,037,855	72.04%	4,578,479	8,520,223	-3,941,744	-3.55%

Algorithm-1: recursive multi-period portfolio optimizer

$[Tree, FV] = \text{RMPO}(x,\, y,\, k_1,\, k_2,\, K,\, d,\, u,\, \pi_d,\, \pi_u,\, r_b,\, \phi,\, \epsilon,\, p,\, m)$

[$L01$] Set $t = (x + y) \times \phi$.

[$L02$] Set $x = x + t$.

[$L03$] If $p < T$

[$L04$] Initialize $Tree$ as an empty tree variable.

[$L05$] Initialize $FV = -\infty$, δ, $Tree_u$, $Tree_d$.

[$L06$] Loop $\delta = -y : \epsilon : x$

[$L07$] If $\delta < 0$, set $x' = x - K' + \delta k_1' - \delta$ and $y' = y + \delta$.

[$L08$] else if $\delta > 0$, set $x' = x - K' - \delta k_2' - \delta$ and $y' = y + \delta$.

[$L09$] else set $x' = x$ and $y' = y$.

Table 9: AVON: asset allocation and cash flow

Year	Stock	Bond	Stock ratio	Cont.	Benefit	NC	NC ratio
2002	1,132,219	296,399	79.25%	56,382	68,004	-11,622	-0.81%
2003	801,006	302,153	72.61%	63,347	70,482	-7,135	-0.65%
2004	1,083,154	354,532	75.34%	71,492	75,354	-3,862	-0.27%
2005	1,341,347	370,825	78.34%	75,361	79,196	-3,835	-0.22%
2006	1,565,453	427,810	78.54%	93,403	81,324	12,079	0.61%
2007	1,683,838	446,958	79.02%	105,149	94,038	11,111	0.52%
2008	1,360,519	495,827	73.29%	112,646	100,908	11,738	0.63%
2009	1,067,809	355,936	75.00%	125,349	111,161	14,188	1.00%
2010	1,463,791	487,930	75.00%	134,681	121,232	13,449	0.69%
2011	1,602,545	534,182	75.00%	139,519	121,745	17,774	0.83%

Table 10: FAPF: asset allocation and cash flow

Year	Stock	Bond	Stock ratio	Cont.	Benefit	NC	NC ratio
2002	66,580	48,611	57.80%	40,186	37,939	2,247	1.95%
2003	79,082	55,498	58.76%	41,481	39,057	2,424	1.80%
2004	90,659	62,358	59.25%	42,904	40,696	2,208	1.44%
2005	110,338	74,643	59.65%	44,883	42,268	2,615	1.41%
2006	118,487	81,420	59.27%	45,906	44,033	1,873	0.94%
2007	126,222	84,704	59.84%	47,603	46,405	1,198	0.57%
2008	89,576	67,120	57.17%	50,783	49,796	987	0.63%
2009	117,526	70,262	62.58%	50,678	54,348	-3,670	-1.95%
2010	126,470	71,135	64.00%	51,267	55,050	-3,783	-1.91%
2011	104,531	87,303	54.49%	53,895	54,919	-1,024	-0.53%

[$L10$] If $(x, y) \in \mathcal{S}$

[$L11$] $[Tree_u, FV_u] = \text{RMPO}(x'r_b,\ y'u,\ k_1,\ k_2,\ K,\ d,\ u,\ \pi_u,\ \pi_d,\ \phi,\ \epsilon,\ p+1)$

[$L12$] $[Tree_d, FV_d] = \text{RMPO}(x'r_b,\ y'd,\ k_1,\ k_2,\ K,\ d,\ u,\ \pi_u,\ \pi_d,\ \phi,\ \epsilon,\ p+1)$

[$L13$] If $FV < \pi_u FV_u + \pi_d FV_d$, set $FV = \pi_u FV_u + \pi_d FV_d$,

[$L14$] $Tree.\delta = \delta$, $Tree.Tree_u = Tree_u$, $Tree.Tree_d = Tree_d$.

[$L15$] End of the loop that began at [$L06$].

[$L16$] else if $p = T$

[$L17$] Solve one-period optimal problem using the method in section 4.2 and return

[$L18$] one-peirod optimal δ, FV, an empty $Tree_u$, and an empty $Tree_d$.

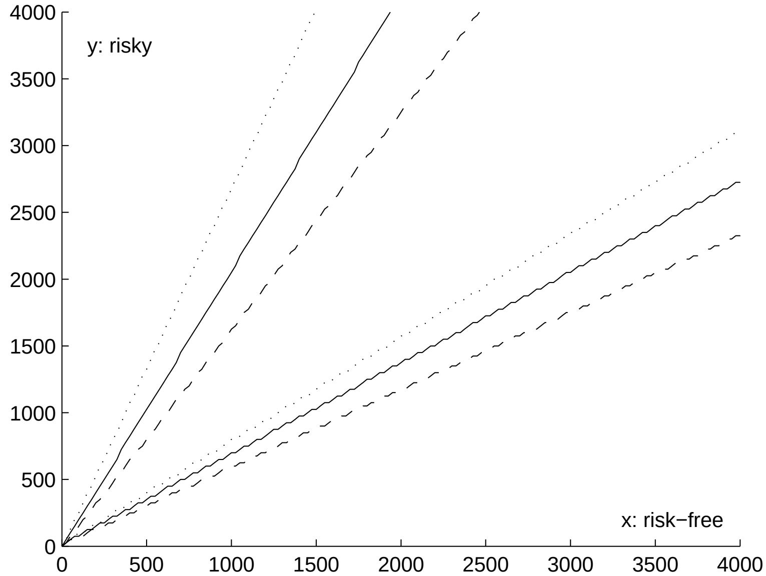

Figure 4: NTR of periodic changes (ϕ) of x: solid (0%), dotted (+5%), dashed (−5%)

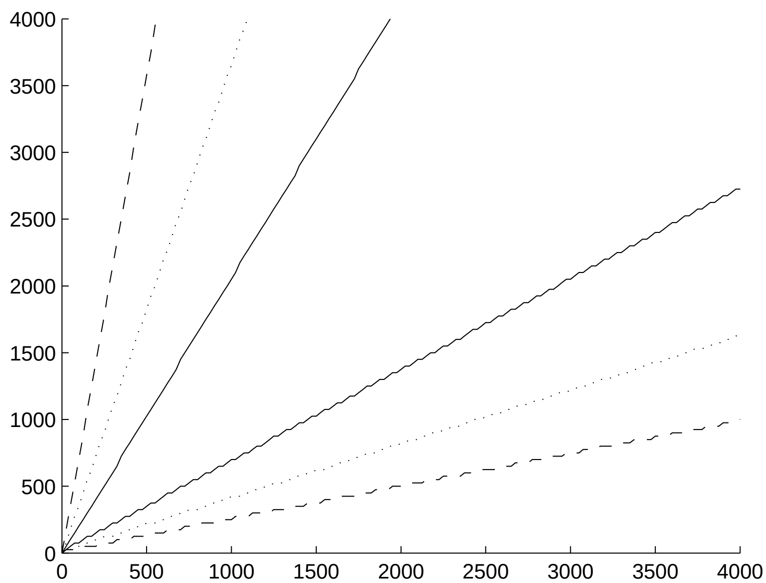

Figure 5: NTR of FPTC with various K: solid (0.1%), dotted (0.5%), dashed (1%)

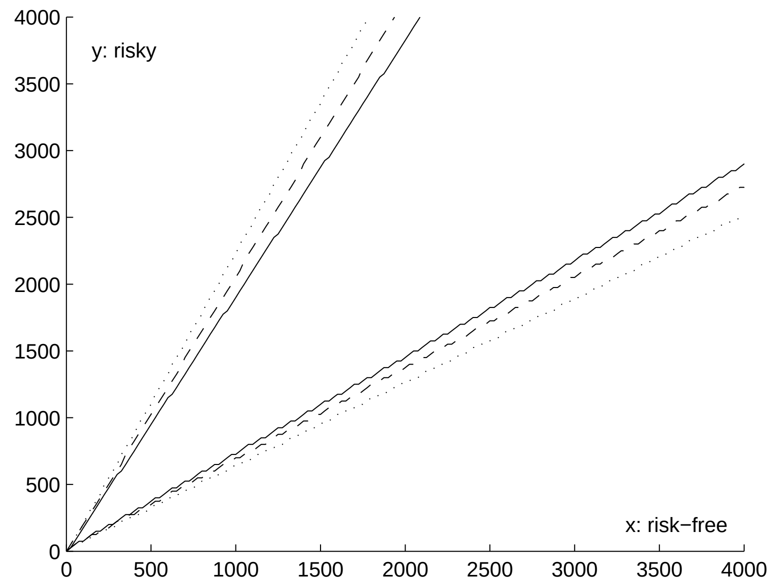

Figure 6: NTR of FPTC with various k_1, k_2: solid (0.1%), dashed (0.5%), dotted (1%)

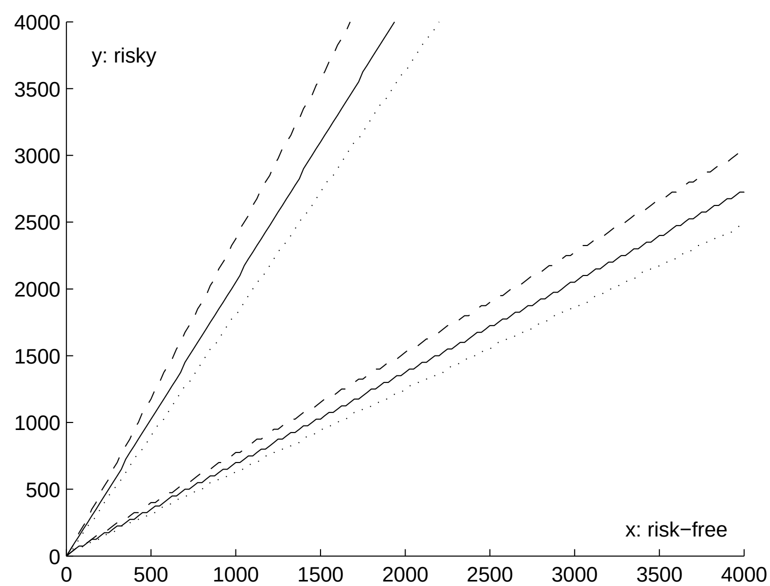

Figure 7: NTR of FPTC with various γ: solid (2), dotted (2.1), dashed (1.9)

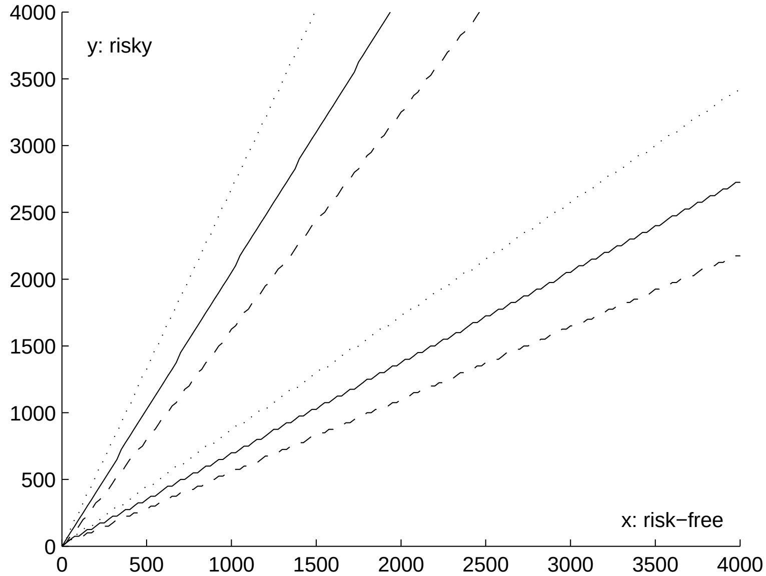

Figure 8: NTR of FPTC with various r_s: solid (1.0771), dotted (1.0810), dashed (1.0732)

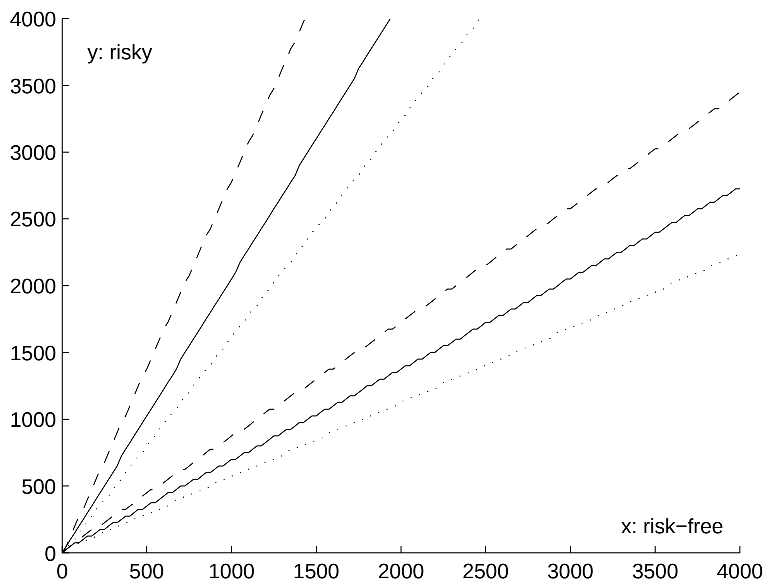

Figure 9: NTR of FPTC with various σ: solid (0.2), dotted (0.21), dashed (0.19)

Table 11: NPS: asset allocation and cash flow

Year	Stock	Bond	Stock ratio	Cont.	Benefit	NC	NC ratio
2004	12,702,300	120,596,100	9.53%	17,143	2,914	14,229	0.0107%
2005	20,394,900	141,482,400	12.6%	18,544	3,584	14,960	0.0092%
2006	21,986,300	164,432,400	11.79%	20,152	4,360	15,792	0.0085%
2007	38,422,600	174,834,000	18.02%	21,670	5,182	16,488	0.0077%
2008	34,263,500	191,124,000	15.2%	22,986	6,180	16,806	0.0075%
2009	49,720,000	215,085,300	18.78%	23,858	7,471	16,387	0.0062%
2010	74,973,200	229,166,000	24.65%	25,285	8,636	16,649	0.0055%
2011	82,061,600	238,071,000	25.63%	27,430	9,819	17,611	0.0055%

Table 12: CALP FPTC allocation

						FPTC optimal risky asset ratio								
ϕ not applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	NA	Spread	Spread-S	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2002	66.69 %	NA	0.27	4.02	19.30 %	92.12	81.17	73.12	85.08	67.00	66.69	58.03	62.04	66.69
2003	69.44 %	NA	0.18	2.66	12.26 %	92.10	81.14	73.06	85.07	69.44	69.44	58.03	62.07	66.06
2004	70.45 %	NA	0.14	2.03	8.19 %	92.10	81.13	73.05	85.06	70.45	70.45	58.04	62.08	66.07
2005	67.88 %	NA	0.15	2.15	8.19 %	92.11	81.16	73.09	85.07	67.88	67.88	58.03	62.05	66.03
2006	71.40 %	NA	0.15	2.13	7.45 %	92.09	81.12	73.03	85.06	71.40	71.40	58.04	62.09	66.09
2007	70.98 %	NA	0.16	2.36	9.33 %	92.10	81.12	73.04	85.06	70.98	70.98	58.04	62.08	66.08
2008	67.12 %	NA	0.23	3.36	23.63 %	92.11	81.17	73.11	85.08	67.12	67.12	58.03	62.05	66.02
2009	60.86 %	NA	0.09	1.36	23.27 %	92.14	81.25	72.20	85.10	67.06	60.86	58.01	60.86	60.86
2010	63.28 %	NA	0.08	1.13	20.60 %	92.13	81.22	72.16	85.09	67.04	63.28	58.02	62.01	63.28
2011	68.75 %	NA	0.07	1.00	17.97 %	92.11	81.15	73.08	85.07	68.75	68.75	58.03	62.06	66.05
Average	67.69 %	NA	0.15	2.22	15.02 %	92.11	81.16	72.89	85.07	68.71	67.69	58.03	61.94	65.32
						FPTC optimal risky asset ratio								
ϕ applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	ϕ	Spread	Spread-S	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2002	66.69 %	-2.98 %	0.27	4.02	19.30 %	89.09	78.11	71.04	83.06	68.74	68.74	56.04	60.08	64.08
2003	69.44 %	-2.65 %	0.18	2.66	12.26 %	89.08	79.09	71.33	83.05	71.33	71.33	56.04	60.10	64.12
2004	70.45 %	-0.87 %	0.14	2.03	8.19 %	91.09	80.11	72.02	84.05	71.07	71.07	57.04	61.09	66.08
2005	67.88 %	0.24 %	0.15	2.15	8.19 %	92.11	81.16	73.10	85.07	67.72	67.72	58.03	62.05	67.01
2006	71.40 %	-0.13 %	0.15	2.13	7.45 %	92.09	81.12	73.03	85.06	71.49	71.49	58.04	62.09	66.09
2007	70.98 %	-0.26 %	0.16	2.36	9.33 %	92.10	81.12	72.02	85.06	71.16	71.16	58.04	61.09	66.09
2008	67.12 %	-0.17 %	0.23	3.36	23.63 %	92.11	81.17	72.09	85.08	67.24	67.24	58.03	62.05	66.02
2009	60.86 %	-0.93 %	0.09	1.36	23.27 %	91.13	80.22	72.19	84.09	67.06	61.43	57.01	61.00	61.43
2010	63.28 %	-1.94 %	0.08	1.13	20.60 %	90.11	79.17	71.12	83.08	66.01	64.53	57.02	61.03	64.53
2011	68.75 %	-2.08 %	0.07	1.00	17.97 %	90.09	79.10	71.01	83.05	70.21	70.21	57.04	60.09	65.08
Average	67.69 %	-1.18 %	0.15	2.22	15.02 %	90.90	80.04	71.90	84.07	69.20	68.49	57.23	61.07	65.05

Table 13: NYSLR FPTC optimal allocation

						FPTC optimal risky asset ratio								
ϕ not applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	NA	Spread	Spread-S	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2003	61.60 %	NA	0.18	2.66	12.26 %	92.14	81.24	72.19	85.10	67.05	61.60	58.01	61.60	61.60
2004	71.61 %	NA	0.14	2.03	8.19 %	92.09	81.11	73.03	85.06	71.61	71.61	58.04	62.09	66.09
2005	80.10 %	NA	0.15	2.15	8.19 %	92.05	81.01	80.10	85.02	80.10	80.10	58.06	62.17	66.23
2006	81.67 %	NA	0.15	2.13	7.45 %	92.05	81.67	81.67	85.01	81.67	81.01	58.07	62.18	66.26
2007	72.88 %	NA	0.16	2.36	9.33 %	92.09	81.10	72.88	85.05	72.88	72.88	58.04	62.10	66.11
2008	84.80 %	NA	0.23	3.36	23.63 %	92.03	84.80	84.80	85.00	84.80	81.08	58.08	61.22	66.31
2009	56.71 %	NA	0.09	1.36	23.27 %	92.16	81.30	72.28	85.12	67.10	56.71	56.71	56.71	56.71
2010	68.30 %	NA	0.08	1.13	20.60 %	92.11	81.15	73.09	85.07	68.30	68.30	58.03	62.06	66.04
2011	72.04 %	NA	0.07	1.00	17.97 %	92.09	81.11	73.02	85.06	72.04	72.04	58.04	62.09	66.10
Average	72.19 %	NA	0.14	2.02	14.54 %	92.09	81.61	75.90	85.05	73.95	71.70	57.90	61.36	64.61

						FPTC optimal risky asset ratio								
ϕ applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	ϕ	Spread	Spread-S	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2003	61.60 %	-4.86 %	0.18	2.66	12.26 %	87.10	77.14	69.07	81.07	64.74	64.74	56.02	59.05	63.03
2004	71.61 %	-3.67 %	0.14	2.03	8.19 %	88.06	78.04	74.33	82.03	74.33	74.33	56.05	60.13	63.18
2005	80.10 %	-2.35 %	0.15	2.15	8.19 %	90.04	82.03	82.03	83.00	82.03	79.06	57.07	60.20	64.29
2006	81.67 %	-2.73 %	0.15	2.13	7.45 %	89.02	83.97	83.97	83.97	82.02	78.12	56.08	60.22	64.32
2007	72.88 %	-2.69 %	0.16	2.36	9.33 %	89.06	79.05	74.89	83.03	74.89	74.89	56.05	60.13	64.17
2008	84.80 %	-5.58 %	0.23	3.36	23.63 %	89.81	88.02	84.12	88.01	79.13	75.28	55.10	58.28	62.43
2009	56.71 %	-5.19 %	0.09	1.36	23.27 %	87.12	76.18	69.16	81.09	64.04	59.82	55.01	59.01	59.82
2010	68.30 %	-4.70 %	0.08	1.13	20.60 %	87.07	77.06	71.67	81.04	71.67	71.67	56.04	59.11	63.14
2011	72.04 %	-3.55 %	0.07	1.00	17.97 %	89.06	78.04	74.69	82.03	74.69	74.69	56.05	60.13	63.18
Average	72.19 %	-3.92 %	0.14	2.02	14.54 %	88.48	79.95	75.99	82.81	74.17	72.51	55.94	59.58	63.06

Table 14: AVON FPTC optimal allocation

						FPTC optimal risky asset ratio								
ϕ not applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	NA	Spread	Spread-S	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2002	79.25 %	NA	0.27	4.02	19.30 %	92.06	81.02	79.25	85.02	79.25	79.25	58.06	62.16	66.22
2003	72.61 %	NA	0.18	2.66	12.26 %	92.09	81.10	72.61	85.05	72.61	72.61	58.04	62.10	66.11
2004	75.34 %	NA	0.14	2.03	8.19 %	92.08	81.07	75.34	85.04	75.34	75.34	58.05	62.12	66.15
2005	78.34 %	NA	0.15	2.15	8.19 %	92.06	81.03	78.34	85.03	78.34	78.34	58.06	62.15	66.20
2006	78.54 %	NA	0.15	2.13	7.45 %	92.06	81.03	78.54	85.03	78.54	78.54	58.06	62.15	66.21
2007	79.02 %	NA	0.16	2.36	9.33 %	92.06	81.02	79.02	85.03	79.02	79.02	58.06	62.16	66.22
2008	73.29 %	NA	0.23	3.36	23.63 %	92.09	81.09	73.29	85.05	73.29	73.29	58.04	62.11	66.12
2009	75.00 %	NA	0.09	1.36	23.27 %	92.08	81.07	75.00	85.04	75.00	75.00	58.05	62.12	66.15
2010	75.00 %	NA	0.08	1.13	20.60 %	92.08	81.07	75.00	85.04	75.00	75.00	58.05	62.12	66.15
2011	75.00 %	NA	0.07	1.00	17.97 %	92.08	81.07	75.00	85.04	75.00	75.00	58.05	62.12	66.15
Average	76.14 %	NA	0.15	2.22	15.02 %	92.07	81.06	76.14	85.04	76.14	76.14	58.05	62.13	66.17

						FPTC optimal risky asset ratio								
ϕ applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	ϕ	Spread	Spread-S	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2002	79.25 %	-0.81 %	0.27	4.02	19.30 %	91.05	80.00	79.90	84.02	79.90	79.90	57.07	61.17	66.23
2003	72.61 %	-0.65 %	0.18	2.66	12.26 %	91.08	80.08	73.09	85.05	73.09	73.09	57.05	61.11	66.12
2004	75.34 %	-0.27 %	0.14	2.03	8.19 %	92.08	81.07	75.54	85.04	75.54	75.54	57.05	61.13	66.16
2005	78.34 %	-0.22 %	0.15	2.15	8.19 %	92.06	81.03	78.51	85.03	78.51	78.51	57.06	61.16	66.21
2006	78.54 %	0.61 %	0.15	2.13	7.45 %	92.06	81.04	78.06	85.03	78.06	78.06	58.06	62.15	67.19
2007	79.02 %	0.52 %	0.16	2.36	9.33 %	92.06	81.03	78.62	85.03	78.62	78.62	58.06	62.15	67.20
2008	73.29 %	0.63 %	0.23	3.36	23.63 %	92.09	81.10	73.00	85.05	72.83	72.83	58.04	62.10	67.10
2009	75.00 %	1.00 %	0.09	1.36	23.27 %	93.09	82.10	74.26	86.05	74.26	74.26	58.05	62.11	67.12
2010	75.00 %	0.69 %	0.08	1.13	20.60 %	93.09	81.08	74.49	86.05	74.49	74.49	58.05	62.12	67.13
2011	75.00 %	0.83 %	0.07	1.00	17.97 %	93.09	81.08	74.38	86.05	74.38	74.38	58.05	62.12	67.12
Average	76.14 %	0.23 %	0.15	2.22	15.02 %	92.18	80.96	75.99	85.24	75.97	75.97	57.65	61.73	66.76

Table 15: FAPF FPTC optimal allocation

						FPTC optimal risky asset ratio								
ϕ not applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	NA	Spread	Spread-S	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2002	57.80 %	NA	0.27	4.02	19.30 %	92.16	81.28	72.26	85.12	67.09	57.80	57.80	57.80	57.80
2003	58.76 %	NA	0.18	2.66	12.26 %	92.15	81.27	72.24	85.11	67.08	58.76	58.00	58.76	58.76
2004	59.25 %	NA	0.14	2.03	8.19 %	92.15	81.27	72.23	85.11	67.08	59.25	58.00	59.25	59.25
2005	59.65 %	NA	0.15	2.15	8.19 %	92.15	81.26	72.22	85.11	67.07	59.65	58.00	59.65	59.65
2006	59.27 %	NA	0.15	2.13	7.45 %	92.15	81.26	72.23	85.11	67.08	59.27	58.00	59.27	59.27
2007	59.84 %	NA	0.16	2.36	9.33 %	92.15	81.26	72.22	85.11	67.07	59.84	58.01	59.84	59.84
2008	57.17 %	NA	0.23	3.36	23.63 %	92.16	81.29	72.27	85.12	67.10	57.17	57.17	57.17	57.17
2009	62.58 %	NA	0.09	1.36	23.27 %	92.14	81.22	72.17	85.10	67.04	62.58	58.01	62.01	62.58
2010	64.00 %	NA	0.08	1.13	20.60 %	92.13	81.21	72.14	85.09	67.03	64.00	58.02	62.02	64.00
2011	54.49 %	NA	0.07	1.00	17.97 %	92.17	81.32	72.32	85.13	67.13	57.04	54.49	54.49	54.49
Average	59.28 %	NA	0.15	2.22	15.02 %	92.15	81.26	72.23	85.11	67.08	59.54	57.55	59.03	59.28

						FPTC optimal risky asset ratio								
ϕ applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	ϕ	Spread	Spread-S	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2002	57.80 %	1.95 %	0.27	4.02	19.30 %	93.17	82.31	74.32	86.13	68.12	57.00	56.69	56.69	56.69
2003	58.76 %	1.80 %	0.18	2.66	12.26 %	93.16	82.30	73.28	86.12	68.10	57.72	57.72	57.72	57.72
2004	59.25 %	1.44 %	0.14	2.03	8.19 %	93.16	82.29	73.27	86.12	68.10	58.41	58.41	58.41	58.41
2005	59.65 %	1.41 %	0.15	2.15	8.19 %	93.16	82.29	73.26	86.12	68.09	58.82	58.00	58.82	58.82
2006	59.27 %	0.94 %	0.15	2.13	7.45 %	93.16	81.27	73.26	86.12	68.09	58.72	58.00	58.72	58.72
2007	59.84 %	0.57 %	0.16	2.36	9.33 %	92.15	81.26	73.25	85.11	68.09	59.50	58.00	59.50	59.50
2008	57.17 %	0.63 %	0.23	3.36	23.63 %	92.16	81.30	73.30	85.12	68.11	56.81	56.81	56.81	56.81
2009	62.58 %	-1.95 %	0.09	1.36	23.27 %	90.12	79.18	71.13	83.08	66.02	63.83	57.02	61.03	63.83
2010	64.00 %	-1.91 %	0.08	1.13	20.60 %	90.11	79.16	71.10	83.07	66.01	65.25	57.02	61.04	65.00
2011	54.49 %	-0.53 %	0.07	1.00	17.97 %	91.17	80.30	72.31	85.13	67.12	56.02	54.78	54.78	54.78
Average	59.28 %	0.43 %	0.15	2.22	15.02 %	92.15	81.17	72.85	85.21	67.59	59.21	57.25	58.35	59.03

Table 16: NPS FPTC optimal allocation

						FPTC optimal risky asset ratio								
ϕ not applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	NA	Spread	Scaled	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2004	9.53 %	NA	0.14	2.03	8.19 %	92.38	80.85	72.11	85.32	67.58	56.66	49.10	45.24	41.33
2005	12.60 %	NA	0.15	2.15	8.19 %	92.37	80.82	73.09	85.31	67.55	56.61	49.09	45.22	41.29
2006	11.79 %	NA	0.15	2.13	7.45 %	92.37	80.83	73.10	85.31	67.56	56.63	49.09	45.23	41.30
2007	18.02 %	NA	0.16	2.36	9.33 %	92.34	80.75	72.98	85.29	67.50	56.54	49.08	45.18	41.24
2008	15.20 %	NA	0.23	3.36	23.63 %	92.35	80.79	73.04	85.30	67.52	56.58	49.08	45.20	41.27
2009	18.78 %	NA	0.09	1.36	23.27 %	92.34	80.74	72.97	85.28	67.49	56.53	49.07	45.18	41.23
2010	24.65 %	NA	0.08	1.13	20.60 %	92.31	80.67	72.86	85.26	67.43	56.44	49.06	46.15	41.17
2011	25.63 %	NA	0.07	1.00	17.97 %	92.31	80.66	72.84	85.25	67.42	56.43	49.06	46.14	41.16
Average	17.03 %	NA	0.13	1.94	14.83 %	92.35	80.76	72.87	85.29	67.51	56.55	49.08	45.44	41.25

						FPTC optimal risky asset ratio								
ϕ applied						Volatility 10%			Volatility 15%			Volatility 20%		
PSP information			World Stock Market			$k_1,\ k_2$			$k_1,\ k_2$			$k_1,\ k_2$		
Year	St. ratio	ϕ	Spread	Scaled	Volatility	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%	0.5%	1.5%	2.5%
2004	9.53 %	0.01 %	0.14	2.03	8.19 %	92.38	80.85	73.14	85.32	67.58	56.66	49.10	45.24	41.33
2005	12.60 %	0.01 %	0.15	2.15	8.19 %	92.37	80.82	73.09	85.31	67.55	56.61	49.09	45.22	41.29
2006	11.79 %	0.01 %	0.15	2.13	7.45 %	92.37	80.83	73.10	85.31	67.56	56.63	49.09	45.23	41.30
2007	18.02 %	0.01 %	0.16	2.36	9.33 %	92.34	80.75	72.99	85.29	67.50	56.54	49.08	45.18	41.24
2008	15.20 %	0.01 %	0.23	3.36	23.63 %	92.35	80.79	73.04	85.30	67.52	56.58	49.08	45.20	41.27
2009	18.78 %	0.01 %	0.09	1.36	23.27 %	92.34	80.74	72.97	85.28	67.49	56.53	49.07	46.19	41.23
2010	24.65 %	0.01 %	0.08	1.13	20.60 %	92.31	80.67	72.86	85.26	67.43	56.44	49.06	46.15	41.17
2011	25.63 %	0.01 %	0.07	1.00	17.97 %	92.31	80.66	72.84	85.25	67.42	56.43	49.06	46.14	41.16
Average	17.03 %	0.01 %	0.13	1.94	14.83 %	92.35	80.76	73.00	85.29	67.51	56.55	49.08	45.57	41.25