An Optimal Job, Consumption/Leisure, and Investment Policy

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Abstract

In this paper we investigate an optimal job, consumption, and investment policy of an economic agent in a continuous and infinite time horizon. The agent's preference is characterized by the Cobb-Douglas utility function whose arguments are consumption and leisure. We use the martingale method to obtain the closed-form solution for the optimal job, consumption, and portfolio policy. We compare the optimal consumption and investment policy with that in the absence of job choice opportunities.

Keywords : job choice, consumption, leisure, portfolio selection, labor income, martingale method.

JEL classification : E21, G11

1 Introduction

We study an optimal job, consumption, and investment policy of an infinitely-lived economic agent whose preference is characterized by the Cobb-Douglas utility function of consumption and leisure. We consider two kinds of jobs one of which provides higher income but lower leisure than the other. We provide the closed-form solution for the optimal job, consumption, and investment policy by using the martingale and duality approach.

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We show that there is a threshold wealth level below which the optimally behaving agent chooses the job providing higher income, but above which he chooses the other job providing higher leisure. This is intuitively appealing since leisure is more important than income as the agent's wealth level gets higher. We show that the agent in our model consumes less(resp. more) when the agent's wealth is below(resp. above) the threshold level than he would if he did not have such job choice opportunities. We also show that the agent in our model takes more risk than he would without the job choice options.

There have been many extensive researches on continuous-time portfolio selection after Merton's pioneering study (Merton [9] and [10]). Bodie, Merton, and Samuelson [1] have studied the effect of the labor-leisure choice on portfolio choice of an economic agent who has flexibility in his labor supply, by using the dynamic programming method. However they did not derive the closed-form solution. In this paper we use the martingale method to derive the closed-form solution. Many papers have considered portfolio selection with a retirement option: for example, Choi and Shim [2], Choi, Shim, and Shin [3], Dybvig and Liu [4], Farhi and Panageas [5], Lim and Shin [8] etc. The retirement in these papers is irreversible in that the agent can not come back to his job after retirement, while the job choices in our model are reversible in that the agent can change the current job at any state and time.

The rest of the paper proceeds as follows. Section 2 sets up the optimization problem. Section 3 provides a solution to the problem and Section 4 investigates properties of the optimal policy.

2 The Model

We consider the continuous-time financial market in an infinite-time horizon. We assume that there are two financial assets in the market: One is a riskless asset and the other is a risky asset. The risk-free interest rate $r > 0$ is assumed to be a constant and the price S_t of the risky asset is governed by the geometric Brownian motion $dS_t/S_t = \mu dt + \sigma dB_t$ for $t \geq 0$, where $(B_t)_{t=0}^{\infty}$ is a standard Brownian motion on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the parameters μ and $\sigma > 0$ are assumed to be constants. We let $\{\mathcal{F}_t\}_{t\geq 0}$ be the augmentation under $\mathbb{P}$ of the natural filtration generated by the standard Brownian motion $(B_t)_{t=0}^{\infty}$.

Let Θ_t denote the job of an economic agent at time t. The job process $\boldsymbol{\Theta} \triangleq (\Theta_t)_{t=0}^{\infty}$

is $\mathcal{F}_t$-adapted. For simplicity, we assume that there are two kinds of jobs, A_0 and A_1. The agent receives constant labor income $Y_i > 0$ and have a leisure rate L_i at each job $A_i,\ i = 0,\ 1$, where

$$0 \le Y_0 < Y_1 \quad \text{and} \quad 0 < L_1 < L_0.$$

Let, $c_t \ge 0$ and π_t denote the consumption rate and the amount of money invested in the risky asset, respectively, at time t. The consumption rate process $\mathbf{c} \triangleq (c_t)_{t=0}^{\infty}$ and the portfolio process $\boldsymbol{\pi} \triangleq (\pi_t)_{t=0}^{\infty}$ are $\mathcal{F}_t$-progressively measurable, $\int_0^t c_s ds < \infty$ for all $t \ge 0$ almost surely(a.s.), and $\int_0^t \pi_s^2 ds < \infty$ for all $t \ge 0$ a.s..

Thus the agent's wealth process $(X_t)_{t=0}^{\infty}$ with $X_0 = x$ evolves according to

$$dX_t = \left[rX_t + (\mu - r)\pi_t - c_t + Y_0 \mathbf{1}_{\{\Theta_t = A_0\}} + Y_1 \mathbf{1}_{\{\Theta_t = A_1\}}\right] dt + \sigma \pi_t dB_t. \tag{2.1}$$

The present value of the future labor income stream is Y_i/r for $\Theta_t = A_i$ where $i = 0, 1$. Since $Y_1/r > Y_0/r$ and the job state process $\boldsymbol{\Theta}$ is chosen endogenously by the agent, we let $X_0 = x > -Y_1/r$ and the agent faces the following wealth constraint:

$$X_t \ge -\frac{Y_1}{r}, \text{ for all } t \ge 0 \text{ a.s..} \tag{2.2}$$

We call a triple of control $(\boldsymbol{\Theta}, \mathbf{c}, \boldsymbol{\pi})$ satisfying the above conditions including (2.2) with $X_0 = x > -Y_1/r$ admissible at x. Let $\mathcal{A}(x)$ be the set of all admissible policies.

We assume that the agent has the Cobb-Douglas utility function $u(c_t, l_t)$, as in Farhi and Panageas [5]:

$$u(c_t, l_t) \triangleq \frac{1}{\alpha} \frac{\left(c_t^{\alpha} l_t^{1-\alpha}\right)^{1-\gamma}}{1-\gamma}, \ 0 < \alpha < 1 \text{ and } 0 < \gamma \ne 1, \tag{2.3}$$

where γ is the agent's coefficient of relative risk aversion, α is a constant, and l_t is the leisure rate at time t. Let $\gamma_1 \triangleq 1 - \alpha(1 - \gamma)$, then $0 < \gamma_1 \ne 1$ and the Cobb-Douglas utility function $u(\cdot, \cdot)$ in (2.3) can be rewritten as

$$u(c_t, l_t) = l_t^{\gamma_1 - \gamma} \frac{c_t^{1-\gamma_1}}{1-\gamma_1}.$$

Remark 2.1. *If $\gamma > 1$, then $\gamma > \gamma_1 > 1$, $\frac{\gamma_1}{1-\gamma_1} < 0$ and $L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} < 0$. If $0 < \gamma < 1$, then $0 < \gamma < \gamma_1 < 1$, $\frac{\gamma_1}{1-\gamma_1} > 0$ and $L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} > 0$. Thus the following inequality always holds:*

$$\frac{\gamma_1}{1-\gamma_1}\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\right) > 0.$$

Problem 2.1. *The agent's optimization problem is to maximize the expected utility*

$$J(x;\boldsymbol{\Theta},\mathbf{c},\boldsymbol{\pi})=\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(L_0^{\gamma_1-\gamma}\frac{c_t^{1-\gamma_1}}{1-\gamma_1}\mathbf{1}_{\{\Theta_t=A_0\}}+L_1^{\gamma_1-\gamma}\frac{c_t^{1-\gamma_1}}{1-\gamma_1}\mathbf{1}_{\{\Theta_t=A_1\}}\right)dt\right],$$

over $(\boldsymbol{\Theta},\mathbf{c},\boldsymbol{\pi})\in\mathcal{A}(x)$, *where* $\rho>0$ *is a subjective discount factor.*

Thus the value function $V(x)$ is given by

$$V(x)=\sup_{(c,\pi,\Theta)\in\mathcal{A}(x)}J(x;\boldsymbol{\Theta},\mathbf{c},\boldsymbol{\pi}).$$

Assumption 2.1. *We assume, as in Farhi and Panageas [5], that*

$$K_1\triangleq r+\frac{\rho-r}{\gamma_1}+\frac{\gamma_1-1}{2\gamma_1^2}\theta^2>0.$$

3 The Solution to the Optimization Problem

We denote the market price of risk and the state price density by θ and H_t, respectively:

$$\theta\triangleq\frac{\mu-r}{\sigma}\quad\text{and}\quad H_t\triangleq e^{-\left(r+\frac{1}{2}\theta^2\right)t-\theta B_t}.$$

For any fixed $T\in[0,\infty)$, we denote the equivalent martingale measure by $\widetilde{\mathbb{P}}^T$:

$$\widetilde{\mathbb{P}}^T(A)=\mathbb{E}\left[e^{-\frac{1}{2}\theta^2T-\theta B_T}\mathbf{1}_A\right],\ \text{for } A\in\mathcal{F}_T.$$

By Girsanov theorem, the new process $\widetilde{B}_t=B_t+\theta t$ is a standard Brownian motion for $t\in[0,T]$ under the measure $\widetilde{\mathbb{P}}^T$. As shown in Proposition 7.4 in Section 1.7 of Karatzas and Shreve [7], there exists a unique probability measure $\widetilde{\mathbb{P}}$ on $\mathcal{F}_\infty$ which agrees with $\widetilde{\mathbb{P}}^T$ on $\mathcal{F}_T$, for $T\in[0,\infty)$, and $\widetilde{B}_t$ is a standard Brownian motion for $t\in[0,\infty)$ under $\widetilde{\mathbb{P}}$. Thus the equation (2.1) can be rewritten as

$$dX_t=\left[rX_t-c_t+Y_0\mathbf{1}_{\{\Theta_t=A_0\}}+Y_1\mathbf{1}_{\{\Theta_t=A_1\}}\right]dt+\sigma\pi_t d\widetilde{B}_t. \tag{3.1}$$

By (2.2) and (3.1), we derive, similarly to Lim and Shin [8], the following budget constraint:

$$\mathbb{E}\left[\int_0^\infty\left(c_t-Y_0\mathbf{1}_{\{\Theta_t=A_0\}}-Y_1\mathbf{1}_{\{\Theta_t=A_1\}}\right)H_t dt\right]\leq x.$$

For a Lagrange multiplier $\lambda>0$, we define a dual value function

$$\begin{aligned}\widetilde{V}(\lambda)+\lambda x=&\sup_{(\boldsymbol{\Theta},\mathbf{c},\boldsymbol{\pi})\in\mathcal{A}(x)}\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(L_0^{\gamma_1-\gamma}\frac{c_t^{1-\gamma_1}}{1-\gamma_1}\mathbf{1}_{\{\Theta_t=A_0\}}+L_1^{\gamma_1-\gamma}\frac{c_t^{1-\gamma_1}}{1-\gamma_1}\mathbf{1}_{\{\Theta_t=A_1\}}\right)dt\right.\\&\left.-\lambda\int_0^\infty\left(c_t-Y_0\mathbf{1}_{\{\Theta_t=A_0\}}-Y_1\mathbf{1}_{\{\Theta_t=A_1\}}\right)H_t dt\right]+\lambda x\\=&\ \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\widetilde{u}_0(\lambda e^{\rho t}H_t)\mathbf{1}_{\{\Theta_t=A_0\}}+\widetilde{u}_1(\lambda e^{\rho t}H_t)\mathbf{1}_{\{\Theta_t=A_1\}}\right)dt\right]+\lambda x,\end{aligned}$$

where

$$\widetilde{u}_i(z) = \sup_{c\geq 0}\left(L_i^{\gamma_1-\gamma}\frac{c^{1-\gamma_1}}{1-\gamma_1} - cz\right) + Y_i z = L_i^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}} + Y_i z,\ i = 0,\ 1.$$

Remark 3.1. *Let $\bar{z}$ be the solution to the algebraic equation $\widetilde{u}_0(z) = \widetilde{u}_1(z)$, then, by Remark 2.1,*

$$\bar{z} = \left(\frac{\frac{\gamma_1}{1-\gamma_1}\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\right)}{Y_1 - Y_0}\right)^{\gamma_1} > 0. \tag{3.2}$$

Thus $\widetilde{V}(\lambda)$ can be rewritten as

$$\widetilde{V}(\lambda) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{\gamma_1}{1-\gamma_1}z_t^{-\frac{1-\gamma_1}{\gamma_1}} + Y_0 z_t\right)\mathbf{1}_{\{\Theta_t = A_0\}} + \left(L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{\gamma_1}{1-\gamma_1}z_t^{-\frac{1-\gamma_1}{\gamma_1}} + Y_1 z_t\right)\mathbf{1}_{\{\Theta_t = A_1\}}\right\}dt\right],$$

where $z_t = \lambda e^{\rho t}H_t = \lambda e^{\left(\rho - r - \frac{1}{2}\theta^2\right)t - \theta B_t}$. Itô's formula to the process z_t implies the stochastic differential equation (SDE) $dz_t/z_t = (\rho - r)dt - \theta dB_t,\ z_0 = \lambda$. Now we define the function

$$\phi(t,z) = \mathbb{E}^{z_t = z}\left[\int_t^\infty e^{-\rho s}\left\{\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{\gamma_1}{1-\gamma_1}z_s^{-\frac{1-\gamma_1}{\gamma_1}} + Y_0 z_s\right)\mathbf{1}_{\{\Theta_s = A_0\}} + \left(L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{\gamma_1}{1-\gamma_1}z_s^{-\frac{1-\gamma_1}{\gamma_1}} + Y_1 z_s\right)\mathbf{1}_{\{\Theta_s = A_1\}}\right\}\right. \tag{3.3}$$

then, by Feynman-Kac formula, the function $\phi(\cdot,\cdot)$ in (3.3) is the solution to the following partial differential equations (PDEs)

$$\begin{cases} \mathcal{L}\phi + e^{-\rho t}\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}} + Y_0 z\right) = 0, & \text{for } \Theta_t = A_0, \\ \mathcal{L}\phi + e^{-\rho t}\left(L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{\gamma_1}{1-\gamma_1}z^{-\frac{1-\gamma_1}{\gamma_1}} + Y_1 z\right) = 0, & \text{for } \Theta_t = A_1, \end{cases} \tag{3.4}$$

where the partial differential operator is defined by

$$\mathcal{L} \triangleq \frac{\partial}{\partial t} + (\rho - r)z\frac{\partial}{\partial z} + \frac{1}{2}\theta^2 z^2\frac{\partial^2}{\partial z^2}.$$

Remark 3.2. *For later use, we consider the quadratic equation*

$$f(n) \triangleq \frac{1}{2}\theta^2 n^2 + \left(\rho - r - \frac{1}{2}\theta^2\right)n - \rho = \frac{1}{2}\theta^2(n - n_+)(n - n_-) = 0,$$

where two roots are $n_+ > 1$ and $n_- < 0$.

Remark 3.3. *Note that*

$$n_- < -\frac{1-\gamma_1}{\gamma_1} < n_+ \tag{3.5}$$

since $f(-(1-\gamma_1)/\gamma_1) = -K_1 < 0$.

Remark 3.4. *If we define a function $g_1(x)$ for $n_- < x < n_+$ as follows:*

$$g_1(x) := -\frac{f(x)}{x - n_-} = -\frac{1}{2}\theta^2(x - n_+) > 0,$$

then $g_1(x)$ is a decreasing function for $n_- < x < n_+$. Thus we have

$$g_1\left(-\frac{1-\gamma_1}{\gamma_1}\right) > g_1(1) > 0 \;\Rightarrow\; -\frac{K_1}{\frac{1-\gamma_1}{\gamma_1} + n_-} > \frac{r}{1-n_-} > 0 \;\Rightarrow\; \frac{\frac{1-\gamma_1}{\gamma_1} + n_-}{K_1} + \frac{1-n_-}{r} > 0. \tag{3.6}$$

Also if we define a a function $g_2(x)$ for $n_- < x < n_+$ as follows:

$$g_2(x) := -\frac{f(x)}{x - n_+} = -\frac{1}{2}\theta^2(x - n_-) < 0,$$

then $g_2(x)$ is also a decreasing function for $n_- < x < n_+$. Thus we have

$$g_2(1) < g_2\left(-\frac{1-\gamma_1}{\gamma_1}\right) < 0 \;\Rightarrow\; \frac{r}{1-n_+} < -\frac{K_1}{\frac{1-\gamma_1}{\gamma_1} + n_+} < 0 \;\Rightarrow\; \frac{\frac{1-\gamma_1}{\gamma_1} + n_+}{K_1} + \frac{1-n_+}{r} > 0. \tag{3.7}$$

Proposition 3.1. *Let*

$$v(z) = \begin{cases} C_1 z^{n_+} + L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{\gamma_1}{(1-\gamma_1)K_1} z^{-\frac{1-\gamma_1}{\gamma_1}} + \frac{Y_0}{r} z, & \text{for } \Theta_t = A_0, \\ D_2 z^{n_-} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{\gamma_1}{(1-\gamma_1)K_1} z^{-\frac{1-\gamma_1}{\gamma_1}} + \frac{Y_1}{r} z, & \text{for } \Theta_t = A_1, \end{cases}$$

where

$$C_1 = \frac{\frac{n_-\gamma_1+1-\gamma_1}{\gamma_1 K_1} + \frac{1-n_-}{r}}{(n_+ - n_-)\bar{z}^{n_+-1}}(Y_1 - Y_0) > 0 \quad \text{and} \quad D_2 = \frac{\frac{n_+\gamma_1+1-\gamma_1}{\gamma_1 K_1} + \frac{1-n_+}{r}}{(n_+ - n_-)\bar{z}^{n_--1}}(Y_1 - Y_0) > 0,$$

then $\phi(t,z) = e^{-\rho t}v(z)$ is the solution to the PDEs (3.4).

Proof. For $\Theta_t = A_0$, we have the PDE

$$\mathcal{L}\phi + e^{-\rho t}\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} + Y_0 z\right) = 0.$$

If we conjecture a trial solution of the form $\phi(t,z) = e^{-\rho t}v(z)$, then we derive the ordinary differential equation (ODE) with respect to z

$$\frac{1}{2}\theta^2 z^2 v''(z) + (\rho - r)zv'(z) - \rho v(z) + L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{\gamma_1}{1-\gamma_1} z^{-\frac{1-\gamma_1}{\gamma_1}} + Y_0 z = 0. \tag{3.8}$$

So the solution of the ODE (3.8) is

$$v(z) = C_1 z^{n_+} + L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{\gamma_1}{(1-\gamma_1)K_1} z^{-\frac{1-\gamma_1}{\gamma_1}} + \frac{Y_0}{r} z.$$

Similarly, for $\Theta_t = A_1$, we obtain the solution

$$v(z) = D_2 z^{n_-} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{\gamma_1}{(1-\gamma_1)K_1} z^{-\frac{1-\gamma_1}{\gamma_1}} + \frac{Y_1}{r} z.$$

Now we use the smooth-pasting condition at $z = \bar{z}$ to determine the constants C_1 and D_2 as follows:

$$C_1 = \frac{\frac{\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\right)(n_-\gamma_1+1-\gamma_1)}{(1-\gamma_1)K_1}\bar{z}^{-\frac{1}{\gamma_1}} + \frac{(1-n_-)(Y_1-Y_0)}{r}}{(n_+ - n_-)\bar{z}^{n_+-1}} = \frac{\frac{n_-\gamma_1+1-\gamma_1}{\gamma_1 K_1} + \frac{1-n_-}{r}}{(n_+ - n_-)\bar{z}^{n_+-1}}(Y_1 - Y_0),$$

and

$$D_2 = \frac{\frac{\left(L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\right)(n_+\gamma_1+1-\gamma_1)}{(1-\gamma_1)K_1}\bar{z}^{-\frac{1}{\gamma_1}} + \frac{(1-n_+)(Y_1-Y_0)}{r}}{(n_+ - n_-)\bar{z}^{n_--1}} = \frac{\frac{n_+\gamma_1+1-\gamma_1}{\gamma_1 K_1} + \frac{1-n_+}{r}}{(n_+ - n_-)\bar{z}^{n_--1}}(Y_1 - Y_0).$$

From (3.6) and (3.7), we see that $C_1 > 0$ and $D_2 > 0$, respectively. $\square$

From (3.3) and Proposition 3.1, it can be shown that $\widetilde{V}(\lambda) = \phi(0, \lambda) = v(\lambda)$. So we use the Legendre transform inverse formula to obtain the value function $V(\cdot)$.

Proposition 3.2. *If $\widetilde{V}(\lambda)$ exists and is differentiable for $\lambda > 0$, then*

$$V(x) = \inf_{\lambda>0}\left(\widetilde{V}(\lambda) + \lambda x\right),$$

for any $x \in (-Y_1/r, \infty)$.

Theorem 3.1. *The value function $V(\cdot)$ is given by*

$$V(x) = \begin{cases} C_1(\lambda_0)^{n_+} + L_0^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{\gamma_1}{(1-\gamma_1)K_1}(\lambda_0)^{-\frac{1-\gamma_1}{\gamma_1}} + \left(x + \frac{Y_0}{r}\right)(\lambda_0), & \text{for } \Theta_t = A_0, \\ D_2(\lambda_1)^{n_-} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{\gamma_1}{(1-\gamma_1)K_1}(\lambda_1)^{-\frac{1-\gamma_1}{\gamma_1}} + \left(x + \frac{Y_1}{r}\right)(\lambda_1), & \text{for } \Theta_t = A_1, \end{cases}$$

where λ_0 and λ_1 are determined from the following algebraic equations

$$x = -n_+C_1(\lambda_0)^{n_+-1} + L_0^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{1}{K_1}(\lambda_0)^{-\frac{1}{\gamma_1}} - \frac{Y_0}{r} \tag{3.9}$$

and

$$x = -n_-D_2(\lambda_1)^{n_--1} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{1}{K_1}(\lambda_1)^{-\frac{1}{\gamma_1}} - \frac{Y_1}{r}, \tag{3.10}$$

respectively.

Remark 3.5. *If we substitute $\bar{z}$ in (3.2) for λ_0 and λ_1 into (3.9) and (3.10), respectively, then we can define the wealth level $\bar{x}$ as*

$$\begin{aligned}\bar{x} &= \left[\frac{\frac{-n_+n_-\gamma_1 - n_+ + n_+\gamma_1}{\gamma_1 K_1} + \frac{n_+n_- - n_+}{r}}{n_+ - n_-} + \frac{L_0^{\frac{\gamma_1-\gamma}{\gamma_1}}}{L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}}\frac{1-\gamma_1}{\gamma_1 K_1}\right](Y_1 - Y_0) - \frac{Y_0}{r} \\ &= \left[\frac{\frac{-n_+n_-\gamma_1 - n_- + n_-\gamma_1}{\gamma_1 K_1} + \frac{n_+n_- - n_-}{r}}{n_+ - n_-} + \frac{L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}}{L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} - L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}}\frac{1-\gamma_1}{\gamma_1 K_1}\right](Y_1 - Y_0) - \frac{Y_1}{r}.\end{aligned}$$

Theorem 3.2. *The optimal policy to Problem 2.1 is* (Θ^*, c^*, π^*) *such that*

$$\Theta_t^* = \begin{cases} A_1, & \text{if } -Y_1/r < X_t < \bar{x}, \\ A_0, & \text{if } X_t \geq \bar{x}, \end{cases} \qquad c_t^* = \begin{cases} L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} \left(z_t^{\lambda_1}\right)^{-\frac{1}{\gamma_1}}, & \text{if } -Y_1/r < X_t < \bar{x}, \\ L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \left(z_t^{\lambda_0}\right)^{-\frac{1}{\gamma_1}}, & \text{if } X_t \geq \bar{x}, \end{cases}$$

and

$$\pi_t^* = \begin{cases} \frac{\theta}{\sigma} \left\{ n_-(n_- - 1) D_2 \left(z_t^{\lambda_1}\right)^{n_- - 1} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{1}{\gamma_1 K_1} \left(z_t^{\lambda_1}\right)^{-\frac{1}{\gamma_1}} \right\}, & \text{if } -Y_1/r < X_t < \bar{x}, \\ \frac{\theta}{\sigma} \left\{ n_+(n_+ - 1) C_1 \left(z_t^{\lambda_0}\right)^{n_+ - 1} + L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{1}{\gamma_1 K_1} \left(z_t^{\lambda_0}\right)^{-\frac{1}{\gamma_1}} \right\}, & \text{if } X_t \geq \bar{x}, \end{cases}$$

where $z_t^{\lambda_0}$ *and* $z_t^{\lambda_1}$ *are determined from the algebraic equations*

$$X_t = -n_+ C_1 \left(z_t^{\lambda_0}\right)^{n_+ - 1} + L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{1}{K_1} \left(z_t^{\lambda_0}\right)^{-\frac{1}{\gamma_1}} - \frac{Y_0}{r} \tag{3.11}$$

and

$$X_t = -n_- D_2 \left(z_t^{\lambda_1}\right)^{n_- - 1} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{1}{K_1} \left(z_t^{\lambda_1}\right)^{-\frac{1}{\gamma_1}} - \frac{Y_1}{r}, \tag{3.12}$$

respectively.

Remark 3.6. *It can be easily shown that* $dX_t/dz_t^{\lambda_0} < 0$ *and* $dX_t/dz_t^{\lambda_1} < 0$ *so that* X_t *in Theorem 3.2 is a decreasing function of* $z_t^{\lambda_0}$ *and* $z_t^{\lambda_1}$*, for* $X_t \geq \bar{x}$ *and for* $-Y_1/r < X_t < \bar{x}$*, respectively.*

4 The Properties of the Solution

We compare the optimal consumption and investment rules with those without the job choice options. If the agent's job were permanently $A_i \in \{A_0, A_1\}$ without job-switching opportunity, then the the optimal consumption/investment strategy, say $(\mathbf{c}^{M_i}, \boldsymbol{\pi}^{M_i})$, under the wealth constraint $X_t \geq -Y_i/r$ with $X_0 = x \geq -Y_i/r$, would be

$$c_t^{M_i} = L_i^{\frac{\gamma_1-\gamma}{\gamma_1}} \left(z_t^{M_i}\right)^{-\frac{1}{\gamma_1}} = K_1 \left(X_t + \frac{Y_i}{r}\right), \quad \pi_t^{M_i} = \frac{\theta}{\sigma\gamma_1} L_i^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{1}{K_1} \left(z_t^{M_i}\right)^{-\frac{1}{\gamma_1}} = \frac{\theta}{\sigma\gamma_1} \left(X_t + \frac{Y_i}{r}\right),$$

where

$$z_t^{M_i} = \frac{L_i^{\gamma_1-\gamma}}{K_1^{\gamma_1}} \left(X_t + \frac{Y_i}{r}\right)^{-\gamma_1}, \ i = 0, 1, \tag{4.1}$$

which can be proved in the same way as in Merton [9] or Karatzas *et al.* [6].

Proposition 4.1. *We have*

$$\begin{cases} c_t^* > c_t^{M_0}, & \text{if } X_t \geq \max(-Y_0/r, \bar{x}) \\ c_t^* < c_t^{M_1}, & \text{if } -Y_1/r < X_t < \bar{x}. \end{cases}$$

Proof. For $X_t \geq \max(-Y_0/r, \bar{x})$, substituting $z_t^{M_0}$ in (4.1) for $z_t^{\lambda_0}$ into X_t in (3.11), then we obtain

$$
\begin{aligned}
X_t\big|_{z_t^{\lambda_0}=z_t^{M_0}} &= -n_+ C_1 \left(z_t^{M_0}\right)^{n_+-1} + L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{1}{K_1} \left(z_t^{M_0}\right)^{-\frac{1}{\gamma_1}} - \frac{Y_0}{r} \\
&= -n_+ C_1 \left(\frac{L_0^{\gamma_1-\gamma}}{K_1^{\gamma_1}} \left(X_t + \frac{Y_0}{r} \right)^{-\gamma_1} \right)^{n_+-1} + X_t + \frac{Y_0}{r} - \frac{Y_0}{r} \\
&< X_t,
\end{aligned}
$$

where the inequality is obtained from the fact $C_1 > 0$ in Proposition 3.1. Since X_t is a decreasing function with respect to z_t, we have

$$
z_t^{M_0} > z_t^{\lambda_0}, \tag{4.2}
$$

and consequently we obtain

$$
c_t^{M_0} = L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \left(z_t^{M_0}\right)^{-\frac{1}{\gamma_1}} < L_0^{\frac{\gamma_1-\gamma}{\gamma_1}} \left(z_t^{\lambda_0}\right)^{-\frac{1}{\gamma_1}} = c_t^*.
$$

For $-Y_1/r < X_t < \bar{x}$, substituting $z_t^{M_1}$ in (4.1) for $z_t^{\lambda_1}$ into X_t in (3.12), then we also obtain

$$
\begin{aligned}
X_t\big|_{z_t^{\lambda_1}=z_t^{M_1}} &= -n_- D_2 \left(z_t^{M_1}\right)^{n_--1} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} \frac{1}{K_1} \left(z_t^{M_1}\right)^{-\frac{1}{\gamma_1}} - \frac{Y_1}{r} \\
&= -n_- D_2 \left(\frac{L_1^{\gamma_1-\gamma}}{K_1^{\gamma_1}} \left(X_t + \frac{Y_1}{r} \right)^{-\gamma_1} \right)^{n_--1} + X_t + \frac{Y_1}{r} - \frac{Y_1}{r} \\
&> X_t,
\end{aligned}
$$

where the inequality is obtained from the facts $n_- < 0$ and $D_2 > 0$ in Proposition 3.1. Since X_t is a decreasing function with respect to z_t, we have

$$
z_t^{M_1} < z_t^{\lambda_1},
$$

and consequently we obtain

$$
c_t^{M_1} = L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} \left(z_t^{M_1}\right)^{-\frac{1}{\gamma_1}} > L_1^{\frac{\gamma_1-\gamma}{\gamma_1}} \left(z_t^{\lambda_1}\right)^{-\frac{1}{\gamma_1}} = c_t^*.
$$

□

Proposition 4.2. *We have*

$$
\begin{cases}
\pi_t^* > \pi_t^{M_0}, & \text{if} \;\; X_t \geq \max(-Y_0/r, \bar{x}) \\
\pi_t^* > \pi_t^{M_1}, & \text{if} \;\; -Y_1/r < X_t < \bar{x}.
\end{cases}
$$

Proof. For $X_t \geq \max(-Y_0/r, \bar{x})$, we obtain

$$\begin{aligned}
\pi_t^* &= \frac{\theta}{\sigma}\left\{n_+(n_+-1)C_1\left(z_t^{\lambda_0}\right)^{n_+-1} + L_0^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{1}{\gamma_1 K_1}\left(z_t^{\lambda_0}\right)^{-\frac{1}{\gamma_1}}\right\}\\
&> \frac{\theta}{\sigma\gamma_1}L_0^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{1}{K_1}\left(z_t^{\lambda_0}\right)^{-\frac{1}{\gamma_1}}\\
&> \frac{\theta}{\sigma\gamma_1}L_0^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{1}{K_1}\left(z_t^{M_0}\right)^{-\frac{1}{\gamma_1}}\\
&= \pi_t^{M_0},
\end{aligned}$$

where the first inequality is obtained from the fact $n_+(n_+-1)C_1 > 0$ and the second inequality is obtained from the inequality (4.2).

For $-Y_1/r < X_t < \bar{x}$, we obtain

$$\begin{aligned}
\pi_t^* &= \frac{\theta}{\sigma}\left\{n_-(n_--1)D_2\left(z_t^{\lambda_1}\right)^{n_--1} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{1}{\gamma_1 K_1}\left(z_t^{\lambda_1}\right)^{-\frac{1}{\gamma_1}}\right\}\\
&= \frac{\theta}{\sigma\gamma_1}\left\{\gamma_1 n_-(n_--1)D_2\left(z_t^{\lambda_1}\right)^{n_--1} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{1}{K_1}\left(z_t^{\lambda_1}\right)^{-\frac{1}{\gamma_1}}\right\}\\
&= \frac{\theta}{\sigma\gamma_1}\left\{-n_-D_2\left(z_t^{\lambda_1}\right)^{n_--1} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{1}{K_1}\left(z_t^{\lambda_1}\right)^{-\frac{1}{\gamma_1}} + n_-\big(1+\gamma_1(n_--1)\big)D_2\left(z_t^{\lambda_1}\right)^{n_--1}\right\}\\
&> \frac{\theta}{\sigma\gamma_1}\left\{-n_-D_2\left(z_t^{\lambda_1}\right)^{n_--1} + L_1^{\frac{\gamma_1-\gamma}{\gamma_1}}\frac{1}{K_1}\left(z_t^{\lambda_1}\right)^{-\frac{1}{\gamma_1}}\right\}\\
&= \frac{\theta}{\sigma\gamma_1}\left(X_t + \frac{Y_1}{r}\right)\\
&= \pi_t^{M_1},
\end{aligned}$$

where the first inequality comes from (3.5) and the fourth equality from (3.12). □

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